Topics

• DTFT and Frequency Response
• Ideal Sampling and Reconstruction
• DFT and FFT
Discrete Time Fourier Transform

\[ X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n} d\omega \]

- Important Properties:
  - Periodicity by \(2\pi\)!
  - Linearity
  - Symmetries (Magnitude, angle, real part, imaginary part)
  - Time shift and modulation
  - Product of signals and convolution
  - Parseval’s Relation
Frequency Response

- For any **stable** LSI system: \( H_d(\omega) = H(z)|_{z=e^{j\omega}} \)

- What is the physical interpretation of this?
  - The DTFT is the z-transform evaluated along the unit circle!

- Why is the frequency response nice to use in addition to the z-transform?
  - \( e^{j\omega} \) is an *eigenfunction* of LSI systems
    - \( h[n] \star A e^{j\omega_0} = \lambda A e^{j\omega_0} = AH_d(\omega_0)e^{j\omega_0} \)
  - By extension:
    - \( x[n] = \cos(\omega_0 n + \theta) \rightarrow y[n] = |H_d(\omega_0)| \cos(\omega_0 n + \theta + \angle H_d(\omega_0)) \)
Magnitude and Phase Response

- Very similar to ECE 210
- Frequency response, and all DTFTs for that matter, are $2\pi$ periodic
- Magnitude response is fairly straightforward
  - Take the magnitude of the frequency response, remembering that $|e^{j\omega}| = 1$
- For phase response:
  - Phase is “contained” in $e^{j\omega}$ terms
  - Remember that cosine and sine introduce sign changes in the phase
    - When a cosine or sine changes phase, we have a contribution of $\pm \pi$ phase.
  - Limit your domain from $-\pi$ to $\pi$.
- For **real-valued** signals and systems:
  - Magnitude response is even-symmetric
  - Phase response is odd-symmetric
DTFT Exercise 1

- Let our signal be

\[ h[n] = \{1, 2, 1\}. \]

a) Compute the DTFT of \( h[n] \).
b) Plot the magnitude response.
c) Plot the phase response.
DTFT Exercise 2

- Suppose we have a new system defined by a real-valued impulse response $h[n]$ with corresponding DTFT $H_d(\omega)$. We also know the following about the magnitude and phase responses:

$$|H_d(\omega)| = \begin{cases} 
1, & \pi \leq \omega < \frac{3\pi}{2} \\
2, & \frac{3\pi}{2} \leq \omega < 2\pi 
\end{cases}$$

$$\angle H_d(\omega) = \begin{cases} 
-\frac{\pi}{4}, & \pi \leq \omega < \frac{3\pi}{2} \\
\frac{\pi}{4}, & \frac{3\pi}{2} \leq \omega < 2\pi 
\end{cases}$$

a) Plot the magnitude response of $H_d(\omega)$ on the interval $-\pi$ to $\pi$.

b) Plot the phase response of $H_d(\omega)$ on the interval $-\pi$ to $\pi$. 
Sinusoidal Response Exercise 1

- We have an LSI system defined by the following LCCDE:

\[ y[n] = x[n] - 2x[n - 1] + x[n - 2]. \]

a) Find \( H(z) \).

b) Find \( H_d(\omega) \).

c) Find the output \( y[n] \) to each of the following inputs:

i. \( x_1[n] = 2 + \cos(\pi n) \)

ii. \( x_2[n] = e^{j\frac{\pi}{4}n} + \sin\left(-\frac{\pi}{2}n\right) \)
Ideal A/D Conversion

- Sampling via an impulse train will yield infinitely many copies of the analog spectrum in the digital frequency domain

\[ X_d(\omega) = \frac{1}{T} \sum_{k \in \mathbb{Z}} X_a \left( \frac{\omega - 2\pi k}{T} \right) \]

- Important relations to recall:
  - Nyquist Sampling Theorem:
    \[ \frac{1}{T} = f_s > 2B = 2f_{\text{max}} \]
  - Relationship between digital \( \omega \) and analog frequencies \( \Omega \):
    \[ \omega = \Omega T \]
Ideal D/A Conversion

- Recall that our DTFT has infinitely many copies of our sampled analog spectrum.

- Ideal D/A conversion requires we perfectly recover only the central copy between $-\pi$ and $\pi$.
  - Digital signal is given notion of continuous-time back with a sampling period $T$.
  - We suppose that we have an ideal low-pass analog filter ("interpolation filter") with cutoff frequency corresponding to $\frac{\pi}{T}$.
Sampling Exercise 1

- Suppose we sampled some analog signal defined by
  \[ x_a(t) = \cos(\Omega_0 t) \]
  with sampling period \( T = \frac{1}{1000} \) s to obtain the digital signal \( x[n] = \cos\left(\frac{\pi}{4} n\right) \). Which of the following are possible values for \( \Omega_0 \)? (There may be more than one!)

  a) \( 250\pi \) rad/s
  b) \( \frac{\pi}{4000} \) rad/s
  c) \( -1750\pi \) rad/s
  d) \( 4250\pi \) rad/s
  e) \( \frac{1}{8} \) rad/s
Sampling Exercise 2

• We have an analog signal $x_a(t)$ with CTFT $X_a(\Omega)$ with maximum frequency $4000\pi$.

![CTFT of $x_a(t)$](image)

For each of the following sampling periods $T$, draw the sampled DTFT spectrum $X_d(\omega)$ on the interval $-3\pi$ to $3\pi$.

a) $T_1 = \frac{1}{8000} \ s$

b) $T_2 = \frac{1}{4000} \ s$

c) $T_3 = \frac{1}{2000} \ s$
Discrete Fourier Transform

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad 0 \leq k \leq N - 1 \]

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \quad 0 \leq n \leq N - 1 \]

- What is the relationship between the DTFT and the DFT?

\[ \omega_k = \frac{2\pi k}{N} \]

\[ \omega_k \in \left\{ 0, \frac{2\pi}{N}, \frac{4\pi}{N}, ..., \frac{2\pi(N-1)}{N} \right\} \]
DFT Properties

- Periodicity by $N$
- Circular shift
- Circular modulation
- Circular convolution

We must amend our DTFT properties with the “circular” term because the DFT is defined over a finite length signal and assumes periodic extension of that finite signal.
Zero-Padding

- We can improve the **resolution** of the DFT simply by adding zeros to the end of the signal.

- This doesn’t change the frequency content of the DTFT!
  - No information/energy is being added.

- Instead, it increases the number of samples the DFT takes of the DTFT.

- This can be used to improve spectral resolution.
Windowing

- Recall that the DFT implies infinite periodic extension of our signal.
- This extension can lead to artifacts known as “spectral leakage”
- Window functions help with these artifacts
  - Rectangular window
  - Hamming window
  - Hanning window
  - Kaiser window
- Windowing is just multiplication in the time domain
  \[ x_w = x[n]w[n] \]
- We care about the main lobe width and side lobe attenuation of these windows.
  - In particular, know the tradeoffs between the rectangular and Hamming windows
Fast Linear Convolution via FFT

- Convolution in the time domain requires $O(n^2)$ operations.
- By convolution theorem, perhaps we can do better in the frequency domain?
- Don’t forget multiplication in DFT domain is circular convolution in time.
- To avoid aliasing, we adopt the following procedure
- Given signal $x$ and filter $h$ of lengths $N$ and $L$, respectively:
  1. Zero-pad $x$ and $h$ to length $N + L - 1$
  2. Take their FFTs
  3. Multiply in frequency domain
  4. Take the inverse FFT

This procedure takes $O(n \log n)$ operations.
DFT Exercise 1

• Surprisingly, we have another signal

\[ x[n] = \cos \left( \frac{\pi}{3} n \right), \ 0 \leq n < 18. \]

a) For which value(s) of \( k \) is the DFT of \( x[n] \), \( X[k] \), largest?

b) Suppose now that we zero-pad our sequence with 72 zeros to obtain \( y[n] \). For which value(s) of \( k \) is \( Y[k] \) largest?