Outline

- LSIC Systems
- BIBO Stability
- Impulse Response and Convolution
- Z-transform
For our system $T$,

- **Linearity:** must satisfy *homogeneity* and *additivity*.
  - Homogeneity: $T(ax) = aT(x)$
  - Additivity: $T(x + z) = T(x) + T(z)$
  - Can be summarized by *superposition*:
    \[ T(ax + bz) = aT(x) + bT(z) \]

- **Shift Invariance:** shifting the input shifts the output by the same amount.
  - If $T(x[n]) = y[n]$, then $T(x[n - n_0]) = y[n - n_0]$.
  - $T(x[n - n_0])$ means we shift our input by $n_0$: we replace every $n$ inside our input arguments with $n - n_0$.
  - $y[n - n_0]$ means we shift our output by $n_0$: we replace every function of $n$ in our output with $f(n) - n_0$.

- **Causality:** output cannot depend on future input values.
BIBO Stability

Three ways to check for BIBO stability:

1. **Absolute summability of the impulse response:**
   \[
   \sum_{n=-\infty}^{\infty} |h[n]| < \infty
   \]

2. **System definition:**
   - Given \( |x[n]| < \alpha \), if \( |T(x[n])| = |y[n]| < \beta < \infty \), \( T \) is BIBO stable
   - In other words, a bounded input yields a bounded output.
   - For example: \( y[n] = x^5[n] + 3 \) vs. \( y[n] = x[n] \ast u[n] \)

3. **Pole-zero plot** (more on this soon)
Convolution

\[ x[n] \ast h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] = \sum_{m=-\infty}^{\infty} h[m]x[n - m] \]

The following statements are equivalent:

- Our system \( T \) is LSI.
- Our system \( T \) can be described by its impulse response \( h[n] \).
- Our system response to an input signal is a convolution.

Also,

- If \( x \) is of length \( L \) and \( h \) is of length \( M \), \( y \) must be of length \( L + M - 1 \).
Impulse Response

Let $x[n]$ be the input to an LSI system with impulse response $h[n]$. Then the system output $y[n]$ is given by:

$$y[n] = x[n] * h[n]$$

By the identity property of convolution, we can find $h[n]$ by passing a Kronecker delta $\delta[n]$ to our system:

$$h[n] = \delta[n] * h[n]$$

For example:

$$y[n] = 2x[n] - 3x[n - 1] + x[n - 2]$$
$$h[n] = 2\delta[n] - 3\delta[n - 1] + \delta[n - 2]$$
Z-transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

- Typically perform inverse z-transform by inspection or by partial fraction decomposition
- Important properties:
  - Multiplication by \( n \): \( nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \)
  - Delay property: \( y[n-k]u[n-k] \leftrightarrow z^{-k}Y(z) \)
- Make sure to note the Region of Convergence (ROC) for your transforms, a Z-transform is not unique without one!
- Convolution theorem: \( y[n] = x[n] * h[n] \xleftrightarrow{zc} X(z)H(z) = Y(z) \)
An LSI system is BIBO stable if its ROC contains the unit circle.

For causal systems, the ROC is anything greater than the outermost pole: \( |z| > |p_{\text{max}}| \)

For a non-causal system, the ROC is anything less than the innermost pole: \( |z| < |p_{\text{min}}| \)

If we sum multiple systems the ROC is the intersection of each system’s ROC.
But what if the ROC is $|z| > 1$ or $|z| < 1$?

- This system is *marginally stable*... though sometimes we just call the system unstable anyway.

- For unstable systems, you are commonly asked to find a bounded input that yields an unbounded output. Few ways to do this:
  - Pick an input that excites the poles of a marginally stable system. This means we pick an input with frequency equal to the angle of the pole. For example, pole at $z = j$ is excited by $e^{j \frac{\pi}{2} n}$ or $\cos \left( \frac{\pi}{2} n \right)$
  - If the system’s impulse response $h[n]$ is not absolutely summable, $u[n]$ works.
  - If the system’s impulse response is unbounded, $\delta[n]$ works.
For the following systems, determine whether it is linear, shift-invariant, and causal.

1. $y[n] = |x[n]|$
2. $y[n] = nx[n]$
3. $y[n] = x[n - 1] + x[|n|]$
4. $y[n] = \log(x[n])$
$y[n] = |x[n]|$ is non-linear, shift-invariant, and causal.

- **Linearity:**
  
  \[
  T(ax) \neq aT(x)
  \]
  
  \[
  T(ax) = |ax[n]| \neq a|x[n]| = aT(x)
  \]
  
  \[\implies \text{Non-linear}\]

- **Shift-invariance:**
  
  \[
  T(x[n - n_0]) \neq y[n - n_0]
  \]
  
  \[
  T(x[n - n_0]) = |x[(n - n_0)]| = |x[(n) - n_0]| = y[n - n_0]
  \]
  
  \[\implies \text{Shift-invariant}\]

- **Causality:** our output time sample is always equal to our input time sample; therefore, this system is causal.
$y[n] = nx[n]$ is linear, shift-varying, and causal.

- **Linearity:**

  $$T(ax + bz) = aT(x) + bT(z)$$
  $$T(ax + bz) = n(ax[n] + bz[n]) = anx[n] + bnz[n] = aT(x) + bT(z)$$
  \[\Rightarrow \text{Linear}\]

- **Shift-invariance:**

  $$T(x[n - n_0]) = y[n - n_0]$$
  $$T(x[n - n_0]) = nx[(n - n_0)] \neq (n - n_0)x[(n) - n_0] = y[n - n_0]$$
  \[\Rightarrow \text{Shift-varying}\]

- **Causality:** our output time sample is always equal to our input time sample; therefore, this system is causal. Don’t for the $n$ out front does not affect causality since it is not part of the input signal’s argument!
LSIC Examples: Part 3

\[ y[n] = x[n-1] + x[|n|] \] is linear, shift-varying, and non-causal.

- **Linearity:**

\[
T(ax + bz) = aT(x) + bT(z)
\]

\[
T(ax + bz) = (ax[n-1] + bz[n-1]) + (ax[|n|] + bz[|n|])
\]

\[
= ax[n-1] + ax[|n|] + bz[n-1] + bz[|n|] = aT(x) + bT(z)
\]

\[ \implies \text{Linear} \]

- **Shift-invariance:**

\[
T(x[n - n_0]) = y[n - n_0]
\]

\[
T(x[n - n_0]) = x[(n - n_0) - 1] + x[|n - n_0|]
\]

\[
\neq x[(n - 1) - n_0] + x[(|n|) - n_0] = y[n - n_0]
\]

\[ \implies \text{Shift-varying} \]

- **Non-causal:** for example, \( y[-3] \) relies on \( x[3] \) in the future.
$y[n] = \log(x[n])$ is non-linear, shift-invariant, and causal.

- **Linearity:**
  
  $$T(ax) \neq aT(x)$$
  
  $$T(ax) = \log(ax[n]) \neq a \log(x[n]) = aT(x)$$
  
  $$\Rightarrow \text{Non-linear}$$

- **Shift-invariance:**
  
  $$T(x[n - n_0]) \neq y[n - n_0]$$
  
  $$T(x[n - n_0]) = \log(x[(n - n_0)]) = \log(x[(n) - n_0]) = y[n - n_0]$$
  
  $$\Rightarrow \text{Shift-invariant}$$

- **Causality:** our output time sample is always equal to our input time sample; therefore, this system is causal.
For each of the following systems defined either by an input-output relationship or impulse response, determine whether the system is BIBO stable or not:

1. \( h[n] = \delta[n] \)
2. \( h[n] = (-\frac{1}{3})^n u[n] \)
3. \( y[n] = x^2[n] + 1 \)
4. \( y[n] = (x[n])^a + b, \ 0 < a, b < c < \infty \)
1. $h[n] = \delta[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
1 \( h[n] = \delta[n] \): \( h[n] \) is absolutely summable; therefore, BIBO stable.

2 \( h[n] = \left(-\frac{1}{3}\right)^n u[n] \): \( h[n] \) is absolutely summable; therefore, BIBO stable.
BIBO Stability Example

1. \(h[n] = \delta[n]\): \(h[n]\) is absolutely summable; therefore, BIBO stable.

2. \(h[n] = (-\frac{1}{3})^n u[n]\): \(h[n]\) is absolutely summable; therefore, BIBO stable.

3. \(y[n] = x^2[n] + 1\): If \(x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty\); therefore, BIBO stable.
1. \( h[n] = \delta[n] \): \( h[n] \) is absolutely summable; therefore, BIBO stable.

2. \( h[n] = (-\frac{1}{3})^n u[n] \): \( h[n] \) is absolutely summable; therefore, BIBO stable.

3. \( y[n] = x^2[n] + 1 \): If \( x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty \); therefore, BIBO stable.

4. \( y[n] = (x[n])^a + b, \ 0 < a, b < c < \infty \):
   \( x[n] < c \implies (x[n])^a < c^a \). Moreover, adding \( b < c < \infty \) will keep our output bounded below \( c^a + b < \infty \); therefore, BIBO stable.
Impulse Response and Convolution Examples

Given \( x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9] \) and \( h[n] = [-1 \ 0 \ 1] \), compute \( y[n] = x[n] \ast h[n] \). Bonus: what does this filter do?

Suppose we have a digital filter \( h[n] \) with an unknown impulse response. We do know the system output to the following two input signals. Determine the impulse response in terms of the two system outputs.

- \( x_1[n] = [2 \ 4 \ 2 \ 4] \rightarrow y_1[n] \)
- \( x_2[n] = [0 \ 2 \ 1 \ 2] \rightarrow y_2[n] \)
Impulse Response and Convolution Examples

Part 1: $x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9]$ and $h[n] = [-1 \ 0 \ 1]$

$y[n] = x[n] * h[n] = [-1 \ -2 \ -2 \ 0 \ -6 \ -6 \ 0 \ 8 \ 9]$
Impulse Response and Convolution Examples

Part 1: \( x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9] \) and \( h[n] = [-1 \ 0 \ 1] \)

\[ y[n] = x[n] * h[n] = [-1 \ -2 \ -2 \ 0 \ -6 \ -6 \ 0 \ 8 \ 9] \]

Part 2:

- \( x_1[n] = [2 \ 4 \ 2 \ 4] \rightarrow y_1[n] \)
- \( x_2[n] = [0 \ 2 \ 1 \ 2] \rightarrow y_2[n] \)

\[ h[n] = h[n] * \delta[n] \]
\[ = h[n] * \left( \frac{1}{2} x_1[n] - x_2[n] \right) \]
\[ = h[n] * \frac{1}{2} x_1[n] - h[n] * x_2[n] \]
\[ = \frac{1}{2} y_1[n] - y_2[n]. \]
Combinations of Systems

Given the following two LSI systems:

\[ h_1[n] = \left( \frac{1}{2} \right)^n u[n], \quad h_2[n] = \left( \frac{1}{3} \right)^n u[n - 1] \]

1. Suppose we pass input \( x[n] \) to system \( h[n] \) which is the connection of \( h_1[n] \) and \( h_2[n] \) in series. Write the output \( y[n] \) in terms of \( x[n] \), \( h_1[n] \) and \( h_2[n] \).

2. Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of \( h[n] \)?

3. Suppose the two systems are now connected in parallel to form \( h[n] \). Now what is the resulting transfer function and impulse response of \( h[n] \)?
Given the following two LSI systems:

\[ h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n - 1] \]

Suppose we pass input \( x[n] \) to system \( h[n] \) is the connection of \( h_1[n] \) and \( h_2[n] \) in series. Write the output \( y[n] \) in terms of \( x[n], h_1[n], \) and \( h_2[n] \).

\[ y[n] = x[n] \ast h_1[n] \ast h_2[n] \]
Given the following two LSI systems:

\[ h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1] \]

Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of \( h[n] \)?

\[ h[n] = h_1[n] * h_2[n] \]
\[ \downarrow \mathcal{Z} \]
\[ H(z) = H_1(z)H_2(z) \]
\[ H_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \]
\[ H_2(z) = \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}} \]

Notice that \( h_2[n] = \left(\frac{1}{3}\right)^n u[n-1] = \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u[n-1] \)
Given the following two LSI systems:

\[ h_1[n] = \left( \frac{1}{2} \right)^n u[n], \quad h_2[n] = \left( \frac{1}{3} \right)^n u[n - 1] \]

Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of \( h[n] \)?

\[
H(z) = \frac{1}{3}z^{-1} \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}
\]

\[
= \frac{K_1}{1 - \frac{1}{2}z^{-1}} + \frac{K_2}{1 - \frac{1}{3}z^{-1}}
\]

PFD \[ \rightarrow K_1 = 2, \quad K_2 = -2 \]

Inspection/Tables \[ \downarrow \]

\[ h[n] = 2 \left( \frac{1}{2} \right)^n u[n] - 2 \left( \frac{1}{3} \right)^n u[n] \]
Combinations of Systems: Part 3

Given the following two LSI systems:

\[
h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n - 1]
\]

Suppose the two systems are now connected in parallel to form \(h[n]\). Now what is the resulting transfer function and impulse response of \(h[n]\)?

\[
h[n] = h_1[n] + h_2[n]
\]

\[
= \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n - 1]
\]

\[
H(z) = H_1(z) + H_2(z)
\]

\[
= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}}
\]
Suppose we have a system response given by $H(z) = \frac{1}{1+z^{-3}}$. Which of the following bounded inputs would cause this system to have an unbounded output? There may be more than one!

1. $\cos\left(\frac{2\pi}{3} n\right) u[n]$
2. $\cos\left(\frac{\pi}{3} n\right) u[n]$
3. $u[n]$
4. $e^{-j\frac{\pi}{3} n} u[n]$
5. $(-1)^n u[n]$
\[ H(z) = \frac{1}{1 + z^{-3}} \]

First, must find the poles of \( H(z) \):

\[ 1 + z^{-3} = 0 \]

\[ z^{-3} = -1 = e^{-j\pi} = e^{-j(\pi + 2\pi k)}, \quad \forall k \in \mathbb{Z} \]

\[ z = e^{j(\frac{\pi}{3} + \frac{2\pi}{3} k)} \]

\[ z_1 = e^{j\frac{\pi}{3}}, \quad z_2 = e^{j\pi}, \quad z_3 = e^{j\frac{5\pi}{3}} \]

Now let’s look at our signals...
Marginal Stability Example

\[ H(z) = \frac{1}{1 + z^{-3}} \]

\[ z_1 = e^{j\frac{\pi}{3}}, \ z_2 = e^{j\pi}, \ z_3 = e^{j\frac{5\pi}{3}} \]

1. \( \cos\left(\frac{2\pi}{3}n\right)u[n] \)
2. \( \cos\left(\frac{\pi}{3}n\right)u[n] \) ✓
3. \( u[n] \)
4. \( e^{-j\frac{\pi}{3}n}u[n] \) ✓
5. \( (-1)^nu[n] \) ✓
Good luck studying and good luck on your exam!