HKN ECE 313 EXAM 1 REVIEW SESSION

Corey Snyder

Sharon Tang

AXIOMS AND PROPERTIES OF PROBABILITY

- Axiom P.1: For any event A, $P(A) \geq 0$
- P.2: If events A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$
- P.3: $P(\Omega) = 1$
- Property p.4: $P(A^C) = 1 - P(A)$
- p.5: $P(A) \leq 1$
- p.6: $P(\emptyset) = 0$
- p.7: If $A \subset B, then\ P(A) \leq P(B)$
- p.8: In general, $P(A \cup B) = P(A) + P(B) - P(AB)$

CARDINALITY OF SETS

- Cardinality of a set is the number of elements in that set

  - Ex: If I pick a card from a deck, the cardinality of spades is 13

- Combinations: order doesn't matter; "n choose k"

  - $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

- Permutations: order **does** matter

  - n!

- Ex: What is the cardinality of a full house?

RANDOM VARIABLES

- Probability Mass Function (pmf):
  - $p_x(u) = P\{X = u\}$
  - $\sum_i p_x(u_i) = 1$
- Expectation (mean) of a random variable:
  - $E[x] = \sum_i u_i \, p_x(u_i)$
- Variance of a random variable:
  - $Var(X) = E[(X - \mu_x)^2] = E[X^2] - (E[X])^2 = \sigma_x^2$
- Standard Deviation:
  - $\sigma_x = \sqrt{Var(X)}$
- What happens if we scale and shift X?
  - $Y = aX + b$
  - $E[Y] = aE[X] + b; Var(Y) = a^2 Var(x)$

CONDITIONAL PROBABILITY, INDEPENDENCE, TOTAL PROBABILITY, BAYES' RULE

- $P(B|A) = \frac{P(AB)}{P(A)} \; if \; P(A) > 0$

  - Finding P(AB) can be challenging, which leads us to…

- Bayes' Rule

  - $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

- Total Probability

- For any event, A, in a sample space that is partitioned by events $E_1, …, E_k$:

  - $P(A) = P(AE_1) + \; … \; + P(AE_k) = P(A|E_1)P(E_1) + \; … \; + P(A|E_k)P(E_k)$

- Independence

  - If $P(AB) = P(A)P(B)$, A and B are independent

  - If $P(ABC) = P(A)P(B)P(C), \; P(AB) = P(A)P(B), \; P(BC) = P(B)P(C) \; and \; P(AC) = P(A)P(C)$

  - And so on…

BERNOULLI AND BINOMIAL DISTRIBUTIONS

- Bernoulli(p): Single trial, probability, p, of success; success = 1
  - pmf: $p(i) = \begin{cases} p & i = 1 \\ 1 - p & i = 0 \end{cases}$
  - Mean: p
  - Variance: $p(1 - p)$
- Binomial(n, p): n independent Bernoulli random variables
  - pmf: $p(i) = \binom{n}{i} p^i (1 - p)^{n-i}$
  - Mean: np
  - Variance: $np(1 - p)$

GEOMETRIC DISTRIBUTION

- Geometric(p): Number of independent Bernoulli trials until a success

  - pmf: $p(i) = (1-p)^{i-1}p$

  - Mean: $\frac{1}{p}$

  - Variance: $\frac{1-p}{p^2}$

- Memoryless Property:

  - $P\{L > i + j \mid L > i\} = P\{L > j\}$

POISSON DISTRIBUTION

- Poisson(λ): $\lambda \geq 0$

  - pmf: $p(i) = \frac{\lambda^i e^{-\lambda}}{i!}$

  - Mean: λ

  - Variance: λ

- Poisson pmf is the limit of the binomial pmf as $n \to \infty$ and $p \to 0$ $such\ that\ np \to \lambda$

MAXIMUM LIKELIHOOD PARAMETER ESTIMATION

- Suppose we have a random variable with a given distribution/pmf that depends on a parameter, θ. By taking trials of the random variable, we can estimate θ by finding the value that maximizes the likelihood of the observed event, $\hat{\theta}_{ML}$.

- There are a few ways we can find $\hat{\theta}_{ML}$

  - Take derivative of provided pmf and set it equal to zero (maximization)

  - Observe the intervals where the likelihood increases and decreases, and find the maximum between these intervals

  - Intuition!

- Intuition Example: If X is drawn at random from numbers 1 through n, with each possibility being equally likely, what is the ML estimator of n?

MARKOV AND CHEBYSHEV INEQUALITIES

- Markov's Inequality

  - $P\{Y \geq c\} \leq \frac{E[Y]}{c}$

- Chebyshev's Inequality

  - $P\{|X - \mu| \geq d\} \leq \frac{\sigma^2}{d^2}$, $commonly\ rewritten\ with\ d = a\sigma, P\{|X - \mu|\} \leq \frac{1}{a^2}$

- Confidence Intervals with Binomial RV: Derivation on Page 51/52 of text (read it!)

  - $P\left\{p \in \left(\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}}\right)\right\} \geq 1 - \frac{1}{a^2}$ where $\hat{p}$ is the estimated value of p, n is the number of trials and $1 - \frac{1}{a^2}$ is our confidence level
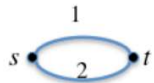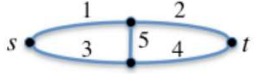
  - $\frac{a}{2\sqrt{n}}$ is referred to as half of the confidence interval

HYPOTHESIS TESTING

- Given the pmf of two hypotheses we want determine which hypothesis is most likely true from a given piece of data
- Maximum Likelihood (ML) Rule
    - $\Lambda(\mathrm{k}) = \frac{p_1(k)}{p_0(k)}$
    - $\Lambda(k) = \begin{cases} > 1 \ declare \ H_1 \ is \ true \\ < 1 \ declare \ H_0 \ is \ true \end{cases}$
- Maximum a Posteriori (MAP) Rule
    - Prior probabilities: $\pi_0 = P(H_0), \pi_1 = P(H_1)$
    - $H_1 true \ if \ \pi_1 p_1(k) > \pi_0 p_0(k),$ same as $\Lambda(\mathrm{k}) = \frac{p_1(k)}{p_0(k)} > \tau \ where \ \tau = \frac{\pi_0}{\pi_1}$
- Probabilities of False Alarm and Miss
    - $p_{false \ alarm} = P(Say \ H_1 \mid H_0 \ is \ true)$
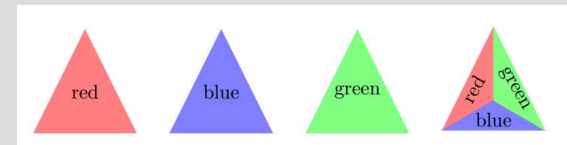   - $p_{miss} = P(Say \ H_0 \mid H_1 \ is \ true)$

UNION BOUND AND ST NETWORKS

- Union Bound

  - $P(A \cup B) \leq P(A) + P(B)$

- ST Networks

  - Given a graph of nodes and independent links with respective failure probabilities, we can calculate the possible carrying capacities of the network and probability of a given capacity, including failure (capacity of zero)

FA15 EXAM 1 Q1

- A tetrahedron has four faces, which are painted as follows: one side all red, one side all blue, one side all green, and one side with red, blue, and green. Assuming all sides are equally likely to be the face that touches the floor:

  - R = {the face that hits the floor has red color}

  - B = {the face that hits the floor has blue color}

  - G = {the face that this the floor has blue color}

  - a) Compute the probabilities: P[R], P[G], P[B]

  - b) Are the events R, G, and B pairwise independent?

  - c) Are the events R, G, B independent?

SP16 EXAM 1 Q1

- Consider three events, A, B, C in a probability space. Let P(AB) = 0.25, P(AB^c) = 0.25. It is known that A and B are independent and that B^c and C are mutually exclusive. Find numerical values for the quantities below, and show your work.

(a) Obtain P(B)

(b) Obtain P(A^{c}B^{c}C)

(c) Obtain P(A^{c}B^{c}C^c)

SP16 EXAM 1 Q2

- Temperature X in degrees F corresponds to temperature (X - 32) *(5/9) in degrees C. For example, 50°F is equivalent to 10°C. According to www.climatestations.com, the standard deviation of the daily temperature (in Chicago, years 1872-2008) is 6°F in July and 12°F in January.

(a) What is the standard deviation of the daily temperature in July, measured in °C? Give the numerical value and show your work.

(b) What is the variance of the daily temperature in July, if temperature is measured in °F?

(c) What is the ratio of the variance of the daily temperature in January to the variance of the daily temperature in July, both measured in °C. Give the numerical answer and briefly explain.

SP14 EXAM 1 Q1

- The Sweet Dreams Cookie Store sells n types of cookies, kept in n separate jars evenly spaced around the edge of a round table. One night, two neighborhood cats, Tom & Jerry, each sneak into the store, choose a jar at random (all jars being equally likely), take a cookie and leave.

    - a) Find the number of ordered pairs of jars that Tom & Jerry can choose to take the cookies from. Assume they can choose the same jar.

    - b) Assume now that Tom comes in first and empties the jar he chooses, so Jerry chooses a different jar at random, again all possibilities being equally likely. Find the number of ordered pairs of jars that Tom & Jerry can choose

    - c) Under the assumption in part b), find the probability that Tom and Jerry pick jars in such a way that there are three or fewer jars between the two they choose. Provide an answer for all values of n with n ≥ 2

FA15 EXAM 1 Q3

- The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson random variable with $\lambda = 2$.

(a) What is the expected number of calls that will arrive during a 10-minute period?

(b) Find the probability that more than three calls will arrive during a 10-minute period.

(c) Find the probability that no calls will arrive during a 10-minute period.

FA13 EXAM 1 Q7

- Consider a regular 8x8 chessboard, which consists of 64 squares in 8 rows and 8 columns.

  - a) How many different rectangles, comprised entirely of chessboard squares, can be drawn on the chessboard? Hint: there are 9 horizontal and 9 vertical lines in the chessboard.

  - b) One of the rectangles you counted in part a) is chosen at random. What is the probability that it is square shaped?

SP14 EXAM 1 Q5

- Two coins, one fair and one coming up heads with probability $\frac{3}{4}$, are in a pocket. One coin is drawn from the pocket, with each possibility having equal probability. The coin drawn is flipped three times.

    - a) What is the probability that heads show up on all three flips?

    - b) Given that a total of two heads show up on the three flips, what is the probability that the coin is fair?

SP16 EXAM 1 Q4

- Suppose X is Binomial(n, p) under hypothesis H_0 and Binomial(n, 1-p) under hypothesis H_1, such that p < 0.5 and n is odd.

(a) Given X=k for some fixed number k, derive the ML rule to decide on the underlying hypothesis. Hint: your decision will depend on k.

(b) What is the probability of missed detection for n=5 and p=1/3?

- The of photons X detected by a particular sensor over a particular time period is assumed to have a Poisson distribution with mean $1 + a^2$, where a is the amplitude of an incident field. It is assumed $a \geq 0$, but otherwise a is unknown.

  - a) Find the maximum likelihood estimate, $\hat{a}_{ML}$, of a for the observation X = 6.

  - b) Find the maximum likelihood estimate, $\hat{a}_{ML}$, of a given that it is observed X = 0.

FA13 EXAM 1 Q4

- A biased coin is tossed repeatedly until a head occurs for the first time. The probability of heads in each toss is $\frac{1}{3}$. Let X denote the number of tosses required until the first heads shows.

  - a) find E[X]

  - b) Find P{X = 7 | X > 5}