HKN ECE 210 Final Exam Review Session

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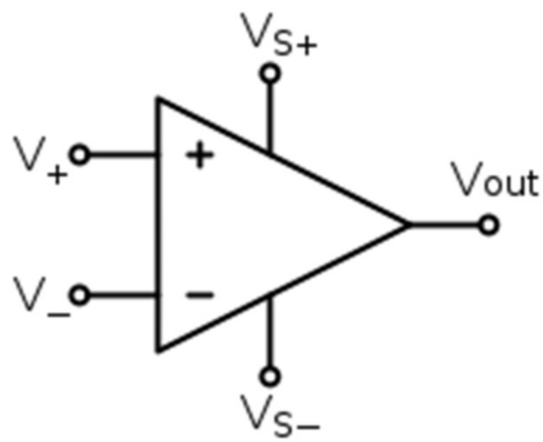
Topics

- Ideal Op-Amps
- Circuit Analysis with Differential Equations
- Average and Maximum Power Transfer
- Fourier Series
- Fourier Transform
- Signal Energy and Bandwidth
- LTI System Response with Fourier Transform
- Modulation, AM, Coherent Demodulation
- Impulse Response and Convolution
- Sampling and Analog Reconstruction
- LTIC and BIBO Stability
- Laplace Transform
- Applications of Laplace Transform

Ideal Op-Amps

- We can make a couple approximations when we work with ideal op-amps

1. The currents i_+ and i_- are both zero.
2. The voltages v_+ and v_- are equal.



Solving RC and RL Circuits with Differential Equations

- Don't forget the fundamentals!
  - $i_C(t) = C\frac{dv_C}{dt}, v_L(t) = L\frac{di_L}{dt}$
- Utilize KVL and KCL to build your differential equations to describe the circuit
- For circuits with one capacitor or inductor, our circuit is described by the following linear ODE

$$\frac{dy}{dt} + ay(t) = bf(t)$$

- $y(t)$ = solution to circuit, e.g. capacitor voltage, inductor current
- $f(t)$ = circuit input, e.g. time varying input like $Ae^{-pt},\ Acos(\omega t)$

Solving RC and RL Circuits with Differential Equations

$$\frac{dy}{dt} + ay(t) = bf(t)$$

- Lots of terms relating various solutions. How do you they all relate?
- Particular Solution: $\frac{dy_p}{dt} + ay_p(t) = bf(t)$ $y_p(t)$ matches the form of $f(t)$
- Homogeneous Solution: $\frac{dy_h}{dt} + ay_h(t) = 0$ $y_h(t)$ looks like Ae^{-pt}
- Zero-Input Response: Same as the Homogenous Solution, except we use a non-zero initial condition
- Zero-State Response: $y_{zs}(t) = y_h(t) + y_p(t)$ such that $y_{zs}(0^-) = 0$
- Transient Response: $y_{tr}(t) =$ components of the response that go to zero as $t \to \infty$, e.g. e^{-pt}
- Steady State Response: $y_{ss}(t) =$ components of the response that remain as $t \to \infty$, e.g. $\cos(\omega t)$, 5
- Full Response: $y(t) = y_{tr}(t) + y_{ss}(t) = y_{zi}(t) + y_{zs}(t)$

Average and Maximum Power Transfer

- Average Power

$$P_{avg} = \frac{1}{2} Re\{VI^*\} = I_{rms} V_{rms}$$

- Available Power (Maximum Power Transfer) from Network: DC/Non-Reactive

$$P_{available} = P_a = \frac{v_T^2}{4R_T}$$

- Maximum power transfer with $R_L = R_T$
- Available Power from Network: Phasor/Reactive

$$P_{available} = P_a = \frac{|V_T|^2}{8Re\{Z_T\}} = \frac{|V_T|^2}{8R_T}$$

- Maximum power transfer is achieved when the load $Z_L = Z_T^*$

LTI System Response to Periodic Inputs

- We have nice ways to find the response of LTI systems to any periodic input:

$$e^{j\omega_0 t} \rightarrow [LTI\ System] \rightarrow H(\omega_0)e^{j\omega_0 t}$$

$$\cos(\omega_0 t) \rightarrow [LTI\ System] \rightarrow |H(\omega_0)| \cos\big(\omega_0 t + \angle H(\omega)\big)$$

$$\sin(\omega_0 t) \rightarrow [LTI\ System] \rightarrow |H(\omega_0)| \sin\big(\omega_0 t + \angle H(\omega)\big)$$

- These results hold for any linear combination of periodic inputs
  - Ex: $8 + 2\cos(2t) + 5\sin(6t)$

Periodic Signals and Fourier Series

- A signal is periodic if and only if there exists a t_0 s.t. $f(t - t_0) = f(t)$ for all t.
- The smallest such t_0 value is referred to as the period $T = \frac{1}{f} = \frac{2\pi}{\omega}$
- Any periodic signal can be expressed as a weighted sum of cosines and sines or complex exponentials. This weighted sum is referred to as the Fourier Series of the signal.

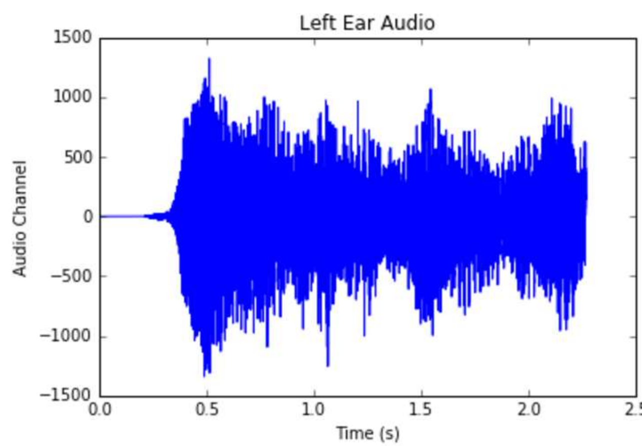
$f(t)$, period $T = \frac{2\pi}{\omega_0}$	Form	Coefficients
$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$	Exponential	$F_n = \frac{1}{T}\int_T f(t)\, e^{-jn\omega_0 t} dt$
$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$	Trigonometric	$a_n = F_n + F_{-n}$ $b_n = j(F_n - F_{-n})$
$\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$	Compact for real $f(t)$	$c_n = 2\|F_n\|$ $\theta_n = \angle F_n$

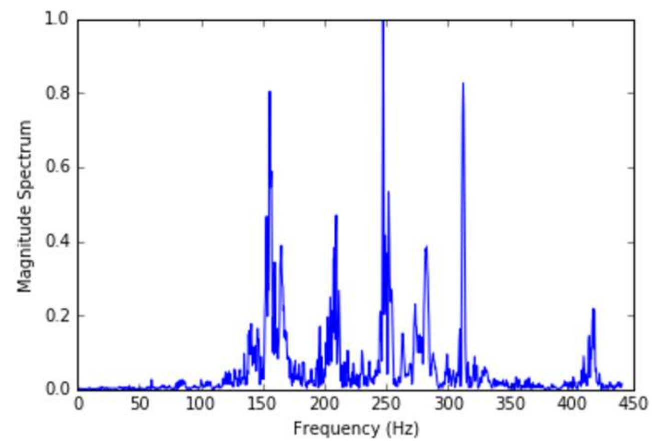
Fourier Series Tips

- Sinusoidal signals will have a finite number of terms in their Fourier Series
- Take advantage of Euler's Identity to quickly get the Fourier Series of sinusoidal signals.
  - Ex: What is the Fourier Series of $\cos^2(4t)$?
- Even symmetric signals will only have non-zero a_n coefficients
- Odd symmetric signals will only have non-zero b_n coefficients
- $\int_T \cos(\omega_1 t)\cos(\omega_2 t) = 0$ if $\omega_1 \neq \omega_2$. Also true for sine!
- $\int_T e^{j\omega_1 t} e^{j\omega_2 t} = 0$ if $\omega_1 \neq -\omega_2$

Fourier Transform

- The Fourier Transform of a signal shows the frequency content of that signal
  - In other words, we can see how much energy is contained at each frequency for that signal
  - ~~This is a big deal!~~
  - This is the biggest deal!
- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$





Important Signals for Fourier Transform

- $rect\left(\frac{t}{T}\right) = \begin{Bmatrix} 1, for\ |t| < \frac{T}{2} \\ 0, for\ |t| > \frac{T}{2} \end{Bmatrix}$
- $u(t) = \begin{Bmatrix} 0, for\ t < 0 \\ 1, for\ t > 0 \end{Bmatrix}$
- $\Delta\left(\frac{t}{T}\right) = \begin{Bmatrix} 1 + \frac{2t}{T}, for \frac{-T}{2} < t < 0 \\ 1 - \frac{2t}{T}, for\ 0 < t < \frac{T}{2} \end{Bmatrix}$
- $sinc(t) = \begin{Bmatrix} \frac{\sin(t)}{t}, for\ t \neq 0 \\ 1, for\ t = 0 \end{Bmatrix}$

Fourier Transform Tips

- Convolution in the time domain is multiplication in the frequency domain
  - $f(t) * g(t) \overset{\mathcal{F}}{\leftrightarrow} F(\omega)G(\omega)$
- Conversely, multiplication in the time domain is convolution in the frequency domain
  - $f(t)g(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}F(\omega) * G(\omega)$
- Scaling your signal can force properties to appear; typically time delay
  - Ex: $e^{-2(t+1)}u(t-1) \rightarrow e^{-4}e^{-2(t-1)}u(t-1)$
- The properties really do matter! Take the time to acquaint yourself with them.
- Remember that the Fourier Transform is linear, so you can express a spectrum as the sum of easier spectra
  - Ex: Staircase function
- Magnitude Spectrum is even symmetric, Phase Spectrum is odd symmetric for real valued signals

Signal Energy and Bandwidth

- $Energy = W = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$
  - Energy signals can be either low-pass or band-pass signals
    - Why not high-pass?
- Bandwidth for Low-pass Signals
  - 3dB BW
    - $\frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2}$
  - r% BW
    - $\frac{1}{2\pi} \int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega = rW$
- Bandwidth for Band-pass signals
  - r% BW
    - $\frac{1}{2\pi} \int_{\omega_l}^{\omega_u} |F(\omega)|^2 \, d\omega = \frac{rW}{2}, \Omega = \omega_u - \omega_l$

LTI System Response using Fourier Transform

- Given the following LTI system:

$$f(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t)$$

- $Y(\omega) = F(\omega)H(\omega)$
- Computationally speaking, and in future courses, we prefer to use Fourier Transforms instead of convolution to evaluate an LTI system
- Why?
  - So much faster: $O(nlogn)\ vs.\ O(n^2)$

Modulation, AM Radio, Coherent Demodulation

- Modulation Property
  - $If\ f(t) \leftrightarrow F(\omega), f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}[F(\omega - \omega_o) + F(w + \omega_o)]$
- In general, for Amplitude Modulation in communications, we modulate with cosine in order to shift our frequency spectrum into different frequency bands
- Coherent Demodulation refers to the process of modulating a signal in order to make it band-pass, then modulating it back to the original low-pass baseband before low-pass filtering in order to recover the original signal
- Envelope detection is the process of Full-wave Rectification (absolute value), then low-pass filtering in order to extract the signal

Impulse Response and Convolution

- Convolution
  - $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$
  - We "flip and shift" one signal and evaluate the integral of the product of the two signals at any value of t (our delay for the shift)
- Representing LTI Systems
  - $y(t) = x(t) * h(t), where\ h(t)\ is\ the\ \boldsymbol{impulse\ response}\ of\ the\ system$
- Impulse Response is the system output to a $\delta(t)$ input
- Graphical convolution helps to visualize the process of flipping and shifting

Helpful Properties for Convolution

- Derivative
  - $h(t) * f(t) = y(t) \rightarrow \frac{d}{dt}h(t) * f(t) = h(t) * \frac{d}{dt}f(t) = \frac{d}{dt}y(t)$
  - Use of Derivative property: Finding the impulse response from the unit-step response
    - $If\ y(t) = u(t) * h(t), then\ \frac{d}{dt}y(t) = \frac{d}{dt}u(t) * h(t) = \delta(t) * h(t) = h(t)$
- Start Point
  - If the two signals have start points at $t_1 and\ t_2$, then the start point of their convolution will be at $t_1 + t_2$
- End Point
  - Similarly for the end points, if the two signals have end points at $t_1 and\ t_2$, then the end point of their convolution will be at $t_1 + t_2$
- Width
  - From the above two properties, we can see that if the two signals have widths W_1 and W_2, then the width of their convolution will be $W_1 + W_2$

The Impulse Function $\boldsymbol{\delta(t)}$

- The impulse function is the limit of $\frac{1}{T} rect\left(\frac{t}{T}\right) as\ T \to 0$
  - Infinitesimal Width
  - Infinite Height
  - Of course, it integrates to 1. $(0 * \infty = 1)$
- Sifting
  - $\int_a^b \delta(t - t_o) f(t) dt = \begin{cases} f(t_0), t_0 \in [a, b] \\ 0, \quad else \end{cases}$
- Sampling
  - $f(t)\delta(t - t_o) = f(t_o)\delta(t - t_o)$
- Unit-step derivative
  - $\frac{du}{dt} = \delta(t)$

Sampling and Analog Reconstruction

- If we have an original analog signal $f(t)$
- Our digital samples of the signal are obtained through sampling property as:
  - $f[n] = f(nT)$ where T is our sampling period; this is Analog to Digital (A/D) conversion
  - This results in infinitely many copies of the original signal's Fourier Transform spaced by $\frac{2\pi}{T}$ and scaled by $\frac{1}{T}$
- We must make sure to satisfy Nyquist Criterion:
  - $T < \frac{1}{2B}$ or $f_s > 2B$
- Following A/D conversion, we perform D/A conversion, then low-pass filter our signal in order to obtain our original signal according to the following relation
- $f(t) = \sum_n f_n sinc\left(\frac{\pi}{T}(t - nT)\right)$
- For a more complete explanation, take ECE 310!

Sampling and Analog Reconstruction (cont'd)

- The complete mathematical relationship between the continuous frequency spectrum and the discrete time (sampled) spectrum in analog frequency

$$F_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(\omega + \frac{2\pi k}{T}\right)$$

- The nuances of this representation will be explored and clarified in ECE 310!

BIBO Stability

- A system is BIBO stable if for any bounded input, we obtain a bounded output
- Two ways to check BIBO stability
- By the definition:
  - $If\ |f(t)| \leq \alpha < \infty, then\ |y(t)| \leq \beta < \infty\ \forall\ t$
- By Absolute Integrability
  - $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

LTIC

- LTIC stands for Linearity Time-Invariance and Causality (sometimes called LSIC, where SI means Shift-Invariance)
- Linearity
  - Satisfy Homogeneity and Additivity
  - Can be summarized by Superposition
    - If $x_1(t) \rightarrow y_1(t)\ and\ x_2(t) \rightarrow\ y_2(t),\ then\ ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
- Shift Invariance
  - If $x(t) \rightarrow y(t)\ then\ x(t - t_o) \rightarrow y(t - t_o)\ \forall\ t_0\ and\ x(t)$
- A system's response can be represented by a convolution **iff** the system is LTI
- Causality
  - Output cannot depend on future input values

Laplace Transform

- The Laplace Transform is another way we can capture the frequency content of a signal.
  - It can be used for stability analysis and initial value circuit problems
- $s = \sigma + j\omega$
- $\widehat{H}(s) = \int_0^\infty h(t)e^{-s}\, dt$
- $h(t) = \beta ig + M\epsilon ss$
  - Use Partial Fraction Expansion and Inspection for the Inverse Laplace Transform!
- Every Laplace transform has a region of convergence.
- If you are instead given the transfer characteristic, the ROC is the right half plane from the rightmost pole.

Applications of the Laplace Transform

- BIBO Stability
  - If the ROC of a system includes the $\sigma = 0$ line, the system is stable.
- Initial Value Circuit Problems
  - A distinct advantage of Laplace Transforms is that they can give the full response of a system, while Fourier Transforms only give us the zero-state response.
  - When transferring a circuit into the s-domain we need to do a few things:
    - Take the Laplace transform of all sources
    - Capacitors go to $\frac{1}{sC}$; inductors go to sL
    - For initial state on a capacitor, place a current source in parallel with the capacitor that points from "-" to "+" with value of $Cv(0^-)$ where $v(0^-)$ is the initial voltage across the capacitor
    - For initial state on an inductor, place a voltage source in series with the inductor such that the current through inductor enters the positive terminal first. This source should have a voltage of $Li(0^-)$ where $i(0^-)$ is the initial current traveling through the inductor

Applications of the Laplace Transform

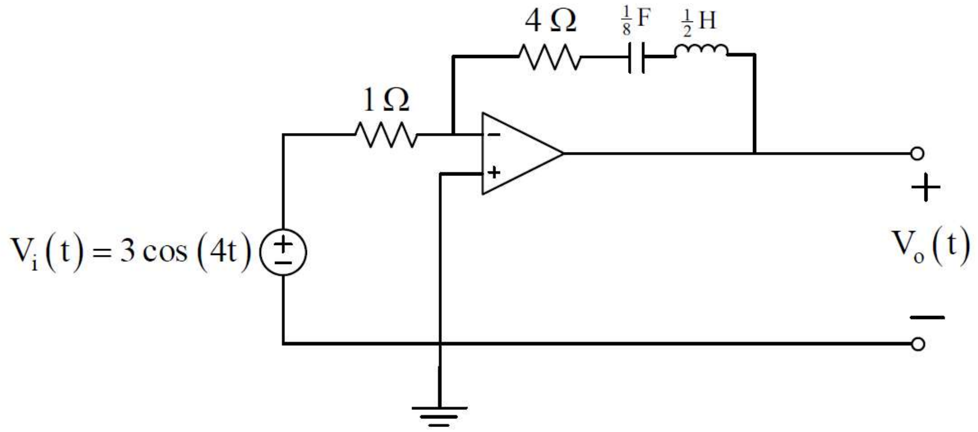
- Characteristic Modes
  - Each pole of the transfer characteristic defines a "characteristic mode"
  - Suppose we have a pole located at c. The characteristic mode is then e^{ct}.
  - The weighted sum of these characteristic modes yields our zero-input response
  - We multiply by t for each repeated pole.
    - Ex: $\hat{H}(s) = \frac{4}{(s+2)^2}$ has characteristic modes $c_1 = e^{-2t}$ and $c_2 = te^{-2}$
      - Zero-input response would be of form $Ae^{-2t} + Bte^{-2t}$
- Solving ODEs
  - The Laplace transform can find the zero-state response for an ODE similar to the Fourier Transform where we simply use the derivative property
  - Can also solve for zero-input response using initial conditions and the characteristic modes. We must solve for the coefficients on each characteristic mode. Thus, need same number of initial conditions as number of poles.

Problem 2 FA15 Exam 2

a. Use the phasor method to calculate the output voltage $V_o(t)$.

b. If a 1000Ω is connected across the output terminals, what is the average power dissipated by the resistor?

c. The op-amp is biased at $\pm 15V$. How would the output voltage change as the input signal frequency is increased or decreased from 4 rad/s?



Problem 1 FA16 Exam 2

- Consider the following circuit with $f(t) = 2\cos(\omega t)$. It is known that for $t > 0$, $i(t) = \cos\left(\frac{1}{2}t\right) + \sin\left(\frac{1}{2}t\right) + Ce^{-\frac{t}{2}}$.

a. Write the ODE that governs this system for $t > 0$ in terms of $L, i(t)$ and ω.

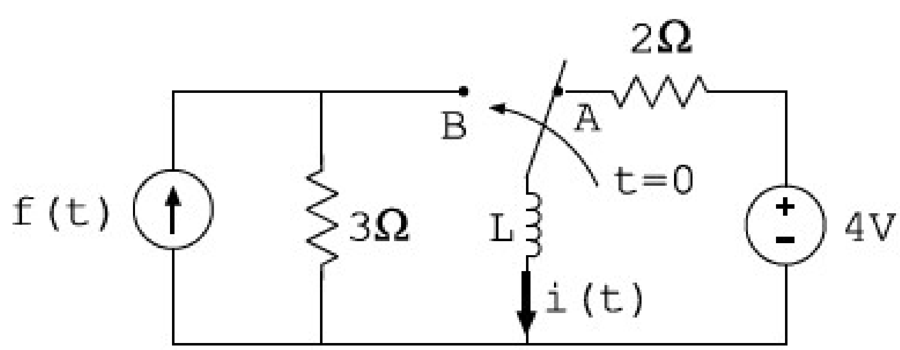
b. Find the value of the inductance L.

c. Find the value of ω.

d. If $i(t)$ is the zero-state response, what is the value of the constant C?

e. If instead $i(t)$ is the full response, what is the value of the constant C?

f. Identify the transient, steady-state, particular and homogeneous components of $i(t)$.



Problem 4 SP15 Exam 2

- For each of the following functions of t, indicate whether or not they are periodic. If periodic, indicate the period, and if not periodic, indicate why.

i. $\sin(\sqrt{2}\pi t) + \cos(\sqrt{3}\pi t)$

ii. $e^{-j\pi} + 2e^{-j}$

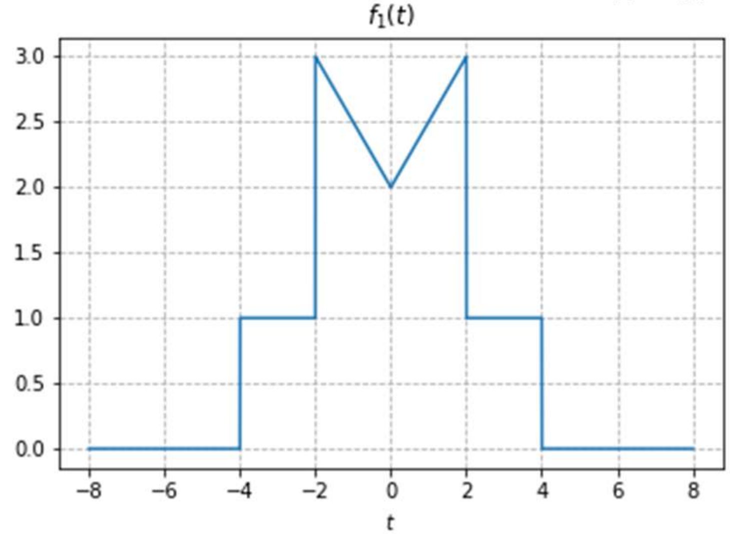
iii. $\sin\left(\frac{\pi}{8}t\right) + 3\sin\left(\frac{5\pi}{2}t\right) + 2\cos\left(\frac{\pi}{4}t\right)$

- Express $f(t) = 2\sin^2\left(t - \frac{\pi}{8}\right)$ as a Fourier series in exponential, trigonometric, and compact forms.

Fourier Transform and Properties Examples

- Find the Fourier Transform of the following signals

a. $f_1(t) =$

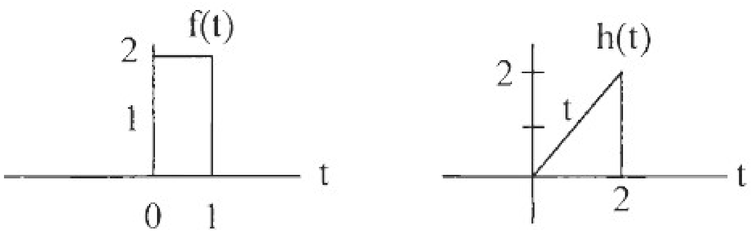


b. $f_2(t) = te^{-(t+2)}u(t-4)$

c. $f_3(t) = \frac{1}{1+j(t-1)} + \frac{1}{1-j(t-1)}$

Problem 3 SP12 Exam 3

- $y(t) = f(t) * h(t)$ where



- Find the following values of $y(t)$:
  - $y(0), y(1), y(2), y(3), y(4)$

- $f(t) = u(t-1)$ is convolved with $h(t) = e^{-3t}u(t)$. Find $y(t) = f(t) * h(t)$ for all values of t.

Problem 4 SP12 Exam 3

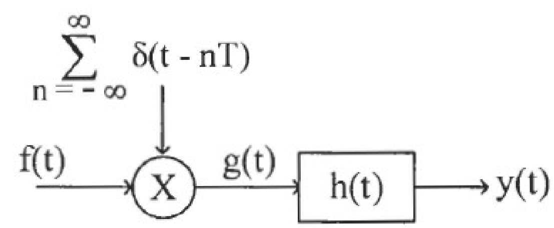
a. Graph $G(\omega)$ between -25 and +25 rad/s.

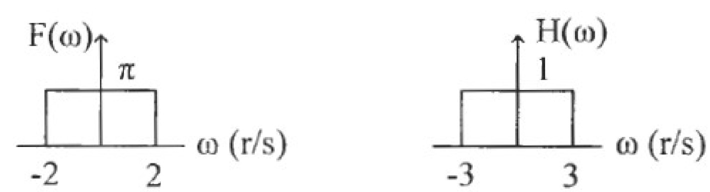
b. Determine $y(t)$ and graph $Y(\omega)$.

c. For the same system, including $F(\omega)$ and $H(\omega)$, T is now unknown. Determine the minimum sampling frequency needed to sample $f(t)$ such that there is no aliasing error in $y(t)$.

Consider the system when

$T = \frac{\pi}{5}$ sec, and $F(\omega)$ and $H(\omega)$ are:





Sample Final Problem 7

a. Draw the equivalent circuit in the s-domain, $t > 0$.

b. Obtain $\hat{V}(s)$, $t > 0$.

c. Obtain $v(t)$, for $t > 0$.

d. Plot $v(t)$ for $t > -1\text{s}$.