HKN ECE 210 Final Exam Review Session

Corey Snyder

Topics

- Circuit Analysis with Differential Equations
- Fourier Series
- Fourier Transform
- Signal Energy and Bandwidth
- LTI System Response with Fourier Transform
- Modulation, AM, Coherent Demodulation
- Impulse Response and Convolution
- Sampling and Analog Reconstruction
- LTIC and BIBO Stability
- Laplace Transform
- Applications of Laplace Transform

Circuit Analysis with Differential Equations

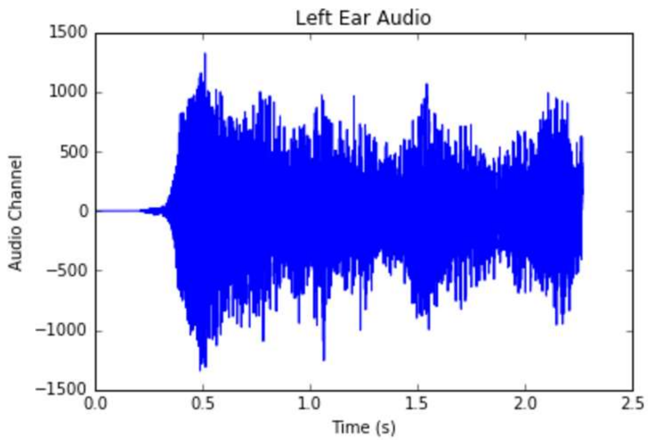
- Given a circuit with time varying inputs and reactive components (capacitors/inductors), we can utilize differential equations in order find the system response
- $i_C = C\frac{dV}{dt}$; $v_L = L\frac{di}{dt}$
- Many different terms, make sure to know the differences!
- Zero-State Response (Particular Solution, zero initial conditions)
- Zero-Input Response (Homogeneous Solution)
- Transient Response – vanishes to zero at positive infinity
- Steady-State Response – persists for all time
- In pairs, these response add up to form the **full response**
  - Full Response = Zero-State Response + Zero-Input Response
  - Full Response = Transient Response + Steady-State Response

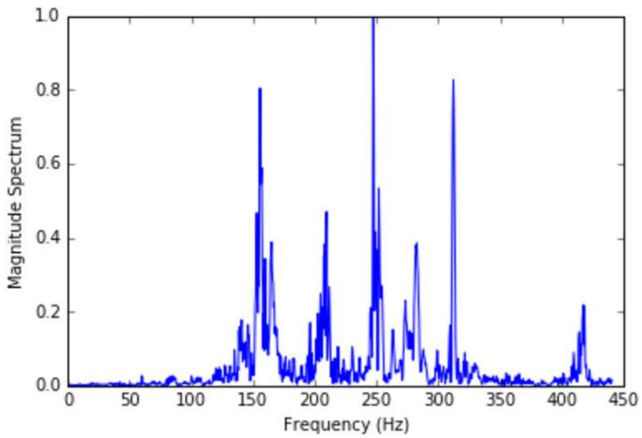
Fourier Series

- Fourier Series allows us to express any periodic signal as the infinite summation of complex exponentials or trigonometric functions
- This is incredibly useful because complex exponentials are eigenfunctions to LTI systems
  - What's an eigenfunction? $f(t) \rightarrow [LTI\ System] \rightarrow \lambda f(t)$ (λ is the eigenvalue for that particular eigenfunction)
- We have three forms: Refer to Table 1 in the Tables Packet for each form
  - Exponential
  - Trigonometric
  - Compact
- Conceptually, Fourier Series is like a discrete version of the Fourier Transform, meaning we only capture specific harmonically related frequencies instead of every frequency
- Total Harmonic Distortion (THD) is the amount of power contained in the harmonics of the fundamental frequency, i.e. for all $n > 1$

Fourier Transform

- The Fourier Transform of a signal shows the frequency content of that signal
  - In other words, we can see how much energy is contained at each frequency for that signal
  - ~~This is a big deal!~~
  - This is the biggest deal!
- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$





Important Signals for Fourier Transform

- $rect\left(\frac{t}{T}\right) = \begin{Bmatrix} 1, for\ |t| < \frac{T}{2} \\ 0, for\ |t| > \frac{T}{2} \end{Bmatrix}$
- $u(t) = \begin{Bmatrix} 0, for\ t < 0 \\ 1, for\ t > 0 \end{Bmatrix}$
- $\Delta\left(\frac{t}{T}\right) = \begin{Bmatrix} 1 + \frac{2t}{T}, for \frac{-T}{2} < t < 0 \\ 1 - \frac{2t}{T}, for\ 0 < t < \frac{T}{2} \end{Bmatrix}$
- $sinc(t) = \begin{Bmatrix} \frac{\sin(t)}{t}, for\ t \neq 0 \\ 1, for\ t = 0 \end{Bmatrix}$

Fourier Transform Tips

- Convolution in the time domain is multiplication in the frequency domain
  - $f(t) * g(t) \overset{\mathcal{F}}{\leftrightarrow} F(\omega)G(\omega)$
- Conversely, multiplication in the time domain is convolution in the frequency domain
  - $f(t)g(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi} F(\omega) * G(\omega)$
- Scaling your signal can force properties to appear; typically time delay
  - Ex: $e^{-2(t+1)}u(t-1) \rightarrow e^{-4}e^{-2(t-1)}u(t-1)$
- The properties really do matter! Take the time to acquaint yourself with them.
- Remember that the Fourier Transform is linear, so you can express a spectrum as the sum of easier spectra
  - Ex: Staircase function
- Magnitude Spectrum is even symmetric, Phase Spectrum is odd symmetric for real valued signals

Signal Energy and Bandwidth

- $Energy = W = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$
  - Energy signals can be either low-pass or band-pass signals
    - Why not high-pass?
- Bandwidth for Low-pass Signals
  - 3dB BW
    - $\frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2}$
  - r% BW
    - $\frac{1}{2\pi} \int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega = rW$
- Bandwidth for Band-pass signals
  - r% BW
    - $\frac{1}{2\pi} \int_{\omega_l}^{\omega_u} |F(\omega)|^2 \, d\omega = \frac{rW}{2}, \Omega = \omega_u - \omega_l$

LTI System Response using Fourier Transform

- Given the following LTI system:

$f(t) \rightarrow$ $H(\omega)$ $\rightarrow y(t)$

- $Y(\omega) = F(\omega)H(\omega)$
- Computationally speaking, and in future courses, we prefer to use Fourier Transforms instead of convolution to evaluate an LTI system
- Why?
  - So much faster: $O(nlogn)\ vs.\ O(n^2)$

Modulation, AM Radio, Coherent Demodulation

- Modulation Property
  - $If\ f(t) \leftrightarrow F(\omega), f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}[F(\omega - \omega_o) + F(w + \omega_o)]$
- In general, for Amplitude Modulation in communications, we modulate with cosine in order to shift our frequency spectrum into different frequency bands
- Coherent Demodulation refers to the process of modulating a signal in order to make it band-pass, then modulating it back to the original low-pass baseband before low-pass filtering in order to recover the original signal
- Envelope detection is the process of Full-wave Rectification (absolute value), then low-pass filtering in order to extract the signal

Impulse Response and Convolution

- Convolution
  - $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$
  - We "flip and shift" one signal and evaluate the integral of the product of the two signals at any value of t (our delay for the shift)
- Representing LTI Systems
  - $y(t) = x(t) * h(t), where\ h(t)\ is\ the\ \boldsymbol{impulse\ response}\ of\ the\ system$
- Impulse Response is the system output to a $\delta(t)$ input
- Graphical convolution helps to visualize the process of flipping and shifting

Helpful Properties for Convolution

- Derivative
  - $h(t) * f(t) = y(t) \rightarrow \frac{d}{dt} h(t) * f(t) = h(t) * \frac{d}{dt} f(t) = \frac{d}{dt} y(t)$
  - Use of Derivative property: Finding the impulse response from the unit-step response
    - $If\ y(t) = u(t) * h(t), then\ \frac{d}{dt} y(t) = \frac{d}{dt} u(t) * h(t) = \delta(t) * h(t) = h(t)$
- Start Point
  - If the two signals have start points at $t_1 and\ t_2$, then the start point of their convolution will be at $t_1 + t_2$
- End Point
  - Similarly for the end points, if the two signals have end points at $t_1 and\ t_2$, then the end point of their convolution will be at $t_1 + t_2$
- Width
  - From the above two properties, we can see that if the two signals have widths W_1 and W_2, then the width of their convolution will be $W_1 + W_2$

The Impulse Function $\boldsymbol{\delta(t)}$

- The impulse function is the limit of $\frac{1}{T} rect\left(\frac{t}{T}\right) as\ T \to 0$
  - Infinitesimal Width
  - Infinite Height
  - Of course, it integrates to 1. $(0 * \infty = 1)$
- Sifting
  - $\int_a^b \delta(t - t_o) f(t) dt = \begin{cases} f(t_0), t_0 \in [a, b] \\ 0, \quad else \end{cases}$
- Sampling
  - $f(t)\delta(t - t_o) = f(t_o)\delta(t - t_o)$
- Unit-step derivative
  - $\frac{du}{dt} = \delta(t)$

Sampling and Analog Reconstruction

- If we have an original analog signal $f(t)$
- Our digital samples of the signal are obtained through sampling property as:
  - $f[n] = f(nT)$ where T is our sampling period; this is Analog to Digital (A/D) conversion
  - This results in infinitely many copies of the original signal's Fourier Transform spaced by $\frac{2\pi}{T}$ and scaled by $\frac{1}{T}$
- We must make sure to satisfy Nyquist Criterion:
  - $T < \frac{1}{2B}$ or $f_s > 2B$
- Following A/D conversion, we perform D/A conversion, then low-pass filter our signal in order to obtain our original signal according to the following relation
- $f(t) = \sum_n f_n sinc(\frac{\pi}{T}(t - nT))$
- For a more complete explanation, take ECE 310!

LTIC

- LTIC stands for Linearity Time-Invariance and Causality (sometimes called LSIC, where SI means Shift-Invariance)
- Linearity
  - Satisfy Homogeneity and Additivity
  - Can be summarized by Superposition
    - If $x_1(t) \rightarrow y_1(t)\ and\ x_2(t) \rightarrow\ y_2(t),\ then\ ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
- Shift Invariance
  - If $x(t) \rightarrow y(t)\ then\ x(t - t_o) \rightarrow y(t - t_o)\ \forall\ t_0\ and\ x(t)$
- Causality
  - Output cannot depend on future input values

BIBO Stability

- A system is BIBO stable if for any bounded input, we obtain a bounded output
- Two ways to check BIBO stability
- By the definition:
  - $If\ |f(t)| \leq \alpha < \infty, then\ |y(t)| \leq \beta < \infty\ \forall\ t$
- By Absolute Integrability
  - $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

Laplace Transform

- The Laplace Transform is another way we can capture the frequency content of a signal.
  - So then why do we need it?
  - It can be used for stability analysis and initial value circuit problems
- $s = \sigma + j\omega$
- $\widehat{H}(s) = \int_0^\infty h(t) e^{-st} dt$
- $h(t) = big\ mess$
  - Use Partial Fraction Expansion and Inspection for the Inverse Laplace Transform!
- Every Laplace transform has a region of convergence.
- If you are instead given the transfer characteristic, the ROC is the right half plane from the rightmost pole.
- s-plane has x-axis of σ and y-axis of $j\omega$

Applications of the Laplace Transform

- BIBO Stability
  - If the ROC of a system includes the $\sigma = 0$ line, the system is stable.
- Initial Value Circuit Problems
  - A distinct advantage of Laplace Transforms is that they can give the full response of a system, while Fourier Transforms only give us the zero-state response.
  - When transferring a circuit into the s-domain we need to do a few things:
    - Take the Laplace transform of all sources
    - Capacitors go to $\frac{1}{sC}$; inductors go to sL
    - For initial state on a capacitor, place a current source in parallel with the capacitor that points from "-" to "+" with value of $Cv(0^-)$ where $v(0^-)$ is the initial voltage across the capacitor
    - For initial state on an inductor, place a voltage source in series with the inductor such that the current through inductor enters the positive terminal first. This source should have a voltage of $Li(0^-)$ where $i(0^-)$ is the initial current traveling through the inductor