Topics Covered

Lectures 23 to 33

1. Wave equations in material media
2. Wave polarization
3. Reflection and transmission
4. Transmission lines

| Condition            | $\beta$       | $\alpha$ | $|\eta|$ | $\tau$ | $\lambda = \frac{2\pi}{\beta}$ | $\delta = \frac{1}{\alpha}$ |
|----------------------|---------------|----------|----------|--------|--------------------------------|-------------------------------|
| Perfect dielectric   | $\sigma = 0$  | $\omega \sqrt{\varepsilon \mu}$ | 0        | $\sqrt{\frac{\mu}{\varepsilon}}$ | 0                              | $\frac{2\pi}{\omega \sqrt{\varepsilon \mu}}$ | $\infty$ |
| Imperfect dielectric | $\frac{\sigma}{\omega \varepsilon} \ll 1$ | $\sim \omega \sqrt{\varepsilon \mu}$ | $\beta \frac{1}{2} \frac{\sigma}{\omega \varepsilon} = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$ | $\sim \sqrt{\frac{\mu}{\varepsilon}}$ | $\sim \frac{\sigma}{2 \omega \varepsilon}$ | $\sim \frac{2\pi}{\omega \sqrt{\varepsilon \mu}}$ | $\frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}}$ |
| Good conductor       | $\frac{\sigma}{\omega \varepsilon} \gg 1$ | $\sim \sqrt{\pi f \mu \sigma}$ | $\sim \sqrt{\pi f \mu \sigma}$ | $\sqrt{\frac{\omega \mu}{\sigma}}$ | 45°                            | $\sim \frac{2\pi}{\sqrt{\pi f \mu \sigma}}$ | $\sim \frac{1}{\sqrt{\pi f \mu \sigma}}$ |
| Perfect conductor    | $\sigma = \infty$ | $\infty$ | $\infty$ | 0       | -                               | 0                             | 0                             |

Figure: Wave properties under different materials
Wave Equations in Material Media

- The propagation velocity of waves in different media is frequency dependent:
  \[ v_p = \frac{\omega}{\beta} \]

- From the wave equation, we can get a general solution for a \( x \)-polarized wave:
  \[ E_x = E_0 e^{\pm \alpha z} e^{\pm \beta z} \]

- Penetration depth, or skin depth:
  \[ \delta = \frac{1}{\alpha} \]

- Wavelength:
  \[ \lambda = \frac{2\pi}{\beta} \]
Wave Properties in Material Media

$$\bar{\gamma} = \alpha + j\beta = \sqrt{j\omega \mu (\sigma + j\omega \varepsilon)}$$

$$\alpha = \omega \sqrt{\frac{\mu \varepsilon}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} - 1 \right)^{1/2}}$$

Attenuation: \(e^{-\alpha z}\)

$$\beta = \omega \sqrt{\frac{\mu \varepsilon}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} + 1 \right)^{1/2}}$$

Velocity: \(v_p = \frac{\omega}{\beta}\)

$$\bar{\eta} = \sqrt{\frac{j\omega \mu}{\sigma + j\omega \varepsilon}}$$
Wave Polarization

- Right-circular polarization: \( \cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \leftrightarrow \hat{x} - j\hat{y} \). The \( x \)-component leads the \( y \)-component. This is the direction your right-hand fingers curl when your thumb is pointed in the \( z \)-direction.

- Left-circular polarization: \( \cos(\omega t)\hat{x} - \sin(\omega t)\hat{y} \leftrightarrow \hat{x} + j\hat{y} \). The \( x \)-component lags the \( y \)-component. This is the direction your left-hand fingers curl when your thumb is pointed in the \( z \)-direction.

\[
\begin{align*}
E_1 &= \hat{x} \cos(\omega t \mp \beta z + \theta_1) \\
E_2 &= \hat{y} \cos(\omega t \mp \beta z + \theta_2) \\
E &= aE_1 + bE_2
\end{align*}
\]

- If \( \theta_1 - \theta_2 = 0 \), \( E \) is linearly polarized. If \( \theta_1 - \theta_2 = \pm \frac{\pi}{2} \), then \( E \) is circularly polarized. Else, it is elliptically polarized.
Reflection and Transmission

- As always, boundary conditions are your friend. The typical problem of normal incidence plane wave travelling from one region to the other has the following boundary conditions in place:

\[
\hat{a}_n = \frac{E_{t1}}{E_{t2}} = \frac{H_{t1}}{H_{t2}} = \frac{J_S\times\hat{a}_n}{D_{n1}-D_{n2} = \rho_s} = \frac{B_{n1}}{B_{n2}}
\]

\[
\hat{a}_n \times (\vec{H}_1 - \vec{H}_2)_t = \vec{J}_S
\]

\[
D_{n1} - D_{n2} = \rho_s
\]
Reflection and Transmission

- If the wave encounters a PEC, it will be completely reflected. Total reflection of a normal planar boundary produces a standing wave which carries no net energy.
- If the wave encounters a region with matched impedance, it is completely transmitted.

\[ \Gamma = \frac{E_1^-}{E_1^+} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \]

\[ \tau = \frac{E_2^+}{E_1^+} = 1 + \Gamma = \frac{2\eta_2}{\eta_1 + \eta_2} \]
Transmission Lines

- At microwave frequencies, lumped elements cannot be used. Transmission lines are a good alternative. If you find this material interesting, look into ECE 447, 451, and 457.
- The transmission line can be modelled as such:

\[
\begin{align*}
V(z + \Delta z) - V(z) &= -L \frac{\partial V}{\partial z} \\
I(z) - I(z + \Delta z) &= C \frac{\partial V}{\partial z}
\end{align*}
\]

\[
\begin{align*}
\therefore \frac{\partial V}{\partial z} &= -L \frac{\partial I}{\partial z} \\
\therefore \frac{\partial I}{\partial z} &= -C \frac{\partial V}{\partial z}
\end{align*}
\]

\[
V = V^+ + V^-
\]

\[
I = \frac{1}{Z_0} (V^+ - V^-) = I^+ + I^-
\]

\[
I^+ = \frac{V^+}{Z_0}, \quad I^- = -\frac{V^-}{Z_0}
\]
Transmission Lines

- The reflection coefficient for a transmission line terminated with a load is shown on the left. The current reflection coefficient is on the right (note the negative sign - this will be important in bounce diagrams).

\[
\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0} \quad \frac{I^-}{I^+} = \frac{-(V^- / Z_0)}{(V^+ / Z_0)} = -\frac{V^-}{V^+} = -\Gamma
\]

- The power relates to the incident, reflected, and transmitted voltages as:

\[
P_{\text{incident}} = V^+ I^+ \\
P_{\text{reflected}} = V^- I^- \\
P_{\text{transmitted}} = V^{++} I^{++}
\]
Bounce Diagrams

Bounce diagrams are a good way of representing voltage and current propagation across a transmission line. The following equation is used to model the voltage at each moment of time. Note that the period, $T$, is 2 times the length of the line divided by the propagation velocity.

\[
V(z, t) = \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v})
+ \tau_g \frac{\Gamma_L}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1)\frac{2\ell}{v})
\]

\[
I(z, t) = \frac{\tau_g}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v})
- \frac{\tau_g}{Z_o} \frac{\Gamma_L}{\Gamma_L \Gamma_g} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1)\frac{2\ell}{v})
\]
Shorted Stubs

- By terminating a transmission line with a short or open stub, we can get different input impedances depending on the frequency and the length of the line.
- The shorted line forms standing voltage waves on the line. At even integers of $\frac{\lambda}{4}$, the line is seen as a short. At odd integers, it is seen as an open.

\[ Z_{in} = jZ_0 \tan \beta l \]

- If $\tan(\beta l) > 0$, shorted
  TL is inductive
- If $\tan(\beta l) < 0$, shorted
  TL is capacitive

\[ V(d, t) = -2|V^+| \sin(\beta d) \sin(\omega t + \theta) \]
\[ I(d, t) = 2Y_0|V^+| \cos(\beta d) \cos(\omega t + \theta) \]

- $V(0, t) = 0$ always (voltage null)
- $I(0, t)$ varies (current maxima)
Open Stubs

- A similar response happens with an open stub.
- At even integers of $\frac{\lambda}{4}$, the line is seen as an open. At odd integers, it is seen as a short.

\[ Y_{in} = jY_0 \tan \beta l \]

If $\tan(\beta l) < 0$, shorted
TL is inductive

If $\tan(\beta l) > 0$, shorted
TL is capacitive

**Same phasor algebra as before with current & voltage reversed!**

\[ I(d, t) = -2Y_0 |V^+| \sin(\beta d) \sin(\omega t + \theta) \]

\[ V(d, t) = 2|V^+| \cos(\beta d) \cos(\omega t + \theta) \]

\[ I(0,t) = 0 \text{ always (current null)} \]

\[ V(0,t) \text{ varies (voltage maxima)} \]
More Notes on Bounce Diagrams

- At steady-state, a transmission line is simply a short.
- If the source has a time-shifted response, for example $f(t) = u(t - 2)$, make sure to shift the bounce diagram up or down accordingly.
- When asked about the voltage or current from a bounce diagram at a certain time or position along the line, draw a vertical line (for position) or a horizontal line (for time) and add the values you 'see' at the point. An example will be discussed in the next.
- Fall 2012 and Spring 2013 exams cover transmission line reflections pretty well.
More notes on Power Conversion

- For problems with a plane-wave incident on an interface, the sum of the transmitted and reflected power equals that of the incident power.

\[ \gamma^2 + \frac{\eta_1}{\eta_2} t^2 = 1 \Rightarrow \frac{\langle s_r \rangle}{\langle s_r \rangle} + \frac{\langle s_t \rangle}{\langle s_t \rangle} = \frac{\langle s_i \rangle}{\langle s_i \rangle} \]

- Go over microwave resonators from Lecture 33. Fall 2012 Problem 3 has a good problem on that topic.