HKN ECE 313 EXAM 2 REVIEW SESSION

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EXAM 2 TOPICS

- Continuous-type Random Variables (mean and variance of CRVs)
- Uniform Distribution
- Exponential Distribution
- Poisson Process
- Linear Scaling of PDFs
- Gaussian Distribution
- ML Parameter Estimation for Continuous Random Variables
- Functions of a random variable
- Failure Rate Functions
- Binary Hypothesis Testing
- Joint CDFs, PMFs, and PDFs
- Independence of Random Variables
- Distributions of sums of random variables

CONTINUOUS-TYPE RANDOM VARIABLES

- Cumulative Distribution Functions (CDFs)
  - Must be nondecreasing
  - Must go to 0 at –infinity and 1 at +infinity
  - Must be right continuous
- $F_X(c) = \int_{-\infty}^{c} f_x(u)du$
- $P\{X = a\} = 0$
- $P\{a < X \leq b\} = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)du$
- $E[X] = \mu_x = \int_{-\infty}^{+\infty} u f_x(u)du$

UNIFORM DISTRIBUTION

- Uniform(a,b):
- $pdf: f(u) = \begin{Bmatrix} \frac{1}{b-a} & a \leq u \leq b \\ 0 & else \end{Bmatrix}$
- $mean: \frac{a+b}{2}$
- $variance: \frac{(b-a)^2}{12}$

EXPONENTIAL DISTRIBUTION

- Exponential(λ): limit of scaled geometric random variables
- $pdf: f(t) = \lambda e^{-\lambda t}\ t \geq 0$
- $mean: \frac{1}{\lambda}$
- $variance: \frac{1}{\lambda^2}$
- Memoryless Property
  - $P\{T \geq s + t \mid T \geq s\} = P\{T \geq t\}\ s, t \geq 0$

POISSON PROCESS

- Poisson process is used to model the number of counts in a time interval. Similar to how the exponential distribution is a limit of the geometric distribution, the Poisson Process is the limited of the Bernoulli process.

- $If\ we\ have\ a\ rate\ \lambda\ and\ time\ interval\ t, the\ number\ of\ counts\ {\sim}Poisson(\lambda t)$

- Disjoint intervals, e.g. 0 to 2s and 2 to 3s, are independent

LINEAR SCALING OF PDFS

- $If\ Y = aX + b$:
- $E[Y] = aE[X] + b$

- $Var(Y) = \ a^2 Var(X)$

- $f_y(v) = \ f_x(\frac{v-b}{a})\frac{1}{a}$

GAUSSIAN DISTRIBUTION

- Gaussian (or Normal) Distribution $\sim N(\mu, \sigma^2)$
  - Standard Gaussian, $\hat{X}: \mu = 0, \sigma = 1$
- $pdf: f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}$
- $mean: \mu$
- $variance: \sigma^2$
- If we standardize our Gaussian, where: $\hat{X} = \frac{X-\mu}{\sigma}$
  - $\Phi(c) = \int_{-\infty}^{c} f(u) d$
  - $Q(c) = 1 - \Phi(c) = \int_{c}^{\infty} f(u) du$

ML PARAMETER ESTIMATION

- Suppose we have a random variable with a given distribution/pdf that depends on a parameter, θ. By taking trials of the random variable, we can estimate θ by finding the value that maximizes the likelihood of the observed event, $\hat{\theta}_{ML}$.
- There are a few ways we can find $\hat{\theta}_{ML}$
  - Take derivative of provided pmf and set it equal to zero (maximization)
  - Observe the intervals where the likelihood increases and decreases, and find the maximum between these intervals
  - Intuition!

FUNCTIONS OF RANDOM VARIABLES

- Suppose Y = g(X), and we want to be able to describe the distribution of Y
- Step 1: Identify the support of X. Sketch the pdf of X and g. Identify the support of Y. Determine whether Y is a Continuous or Discrete RV
- Step 2 (for CRV): Use the definition of the CDF to find the CDF of Y:
  - $F_Y(c) = P\{Y \leq c\} = P\{g(X) \leq c\}$
- Step 2 (for DRV): Find the pmf of Y directly using the definition of the pmf
  - $p_Y(v) = P\{Y = v\} = P\{g(X) = v\}$
- Step 3 (for CRV): Differentiate the CDF of Y in order to find the pdf of Y

GENERATING A RV WITH A SPECIFIED DISTRIBUTION

- We can generate any distribution by applying a function to a uniform distribution
- This function should be the inverse of the CDF of the desired distribution
- Ex: if we want an exponential distribution,
  - $F_X(c) = 1 - e^{-c} = u$; then find $F_X^{-1}(c)$

FAILURE RATE FUNCTIONS

- We can assess the probability of a failure in a system through a failure rate function, h(t).
- $F(t) = 1 - e^{-\int_0^t h(s)ds}$
- Two popular failure rate funcitons:
  - Consistent lifetime
  - "Bath tub"

BINARY HYPOTHESIS TESTING

- Similar to BHT with Discrete RVs
- Maximum Likelihood (ML) Rule
  - $\Lambda(\mathrm{k}) = \frac{f_1(k)}{f_0(k)}$
  - $\Lambda(k) = \begin{cases} > 1 \; declare \; H_1 \; is \; true \\ < 1 \; declare \; H_0 \; is \; true \end{cases}$
- Maximum a Posteriori (MAP) Rule
  - Prior probabilities: $\pi_0 = P(H_0), \pi_1 = P(H_1)$
  - $H_1 true \; if \; \pi_1 f_1(k) > \pi_0 f_0(k)$, same as $\Lambda(\mathrm{k}) = \frac{f_1(k)}{f_0(k)} > \tau \; where \; \tau = \frac{\pi_0}{\pi_1}$
- Probabilities of False Alarm and Miss
  - $p_{false\;alarm} = P(Say\; H_1 \mid H_0 \; is \; true)$
  - $p_{miss} = P(Say\; H_0 \mid H_1 \; is \; true)$
  - $p_e = \pi_1 p_{miss} + \pi_o p_{false\;alarm}$

JOINT CDF, PMF, AND PDF

DISCRETE RANDOM VARIABLES

- Joint CDF
  - $F_{X,Y}(u_o, v_o) = P\{X \leq u_o, Y \leq v_o\}$
- Joint PMF
  - $p_{X,Y}(u_o, v_o) = P\{X = u_o, Y = v_o\}$
- Marginal PMFs
  - $p_X(u) = \sum_j p_{X,Y}(u, v_j)$
  - $p_Y(v) = \sum_i p_{X,Y}(u_i, v)$
- Conditional PMFs
  - $p_{(Y|X)}(v|u_o) = P(Y = v|X = u_o) = \frac{p_{X,Y}(u_o,v)}{p_X(u_o)}$

CONTINUOUS RANDOM VARIABLES

- Joint CDF
  - $F_{X,Y}(u_o, v_o) = P\{X \leq u_o, Y \leq v_o\}$
- Joint PDF
  - $F_{X,Y}(u_o, v_o) = \iint f_{X,Y}(u, v)dvd$
- Marginal PDFs
  - $f_X(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)dv$
  - $f_Y(v) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)du$,
- Conditional PDFs
  - $f_{(Y|X)}(v|u_o) = \frac{f_{X,Y}(u_o,v)}{f_X(u_o)}$

INDEPENDENCE OF JOINT DISTRIBUTIONS

- We can check independence in a joint distribution in a couple ways:

- $f_{X,Y}(u,v) = f_X(u)f_Y(v)$

- The support of $f_{X,Y}(u,v)$ is a *product set*
  - Product set must have the swap property, which is satisfied if:
    - $(a,b)\ and\ (c,d) \in support\left(f_{X,Y}\right),\ and\ then\ (a,d)\ and\ (b,c)\ are\ also \in support(f_{X,Y})$
  - Checking for a product set is only sufficient to prove *dependence*. Saying that the joint pdf is a product set is not sufficient to check independence

DISTRIBUTION OF SUMS OF RANDOM VARIABLES

- Suppose we want to find the distribution from the sum of two independent random variables where $Z = X + Y$

- The pdf or pmf of Z is the *convolution* of the two pdfs/pmfs

FA15 PROBLEM 5

5. **[18 points]** Suppose we are testing if a transistor in a circuit is working properly or not. The voltage we observe at the output is a normal random variable X. If the transistor is working, X is distributed according to $N(1,1)$. If the transistor is not working, then X is distributed according to $N(-1,1)$. There is an 50% chance that the transistor is working. You can express the answers for this problem in terms of Φ and Q.

(a) (6 points) Find $P\{X \leq 1 | \text{transistor is working}\}$.

(b) (6 points) Find $P\{X \geq 1\}$.

(c) (6 points) Obtain the unconditional pdf of X, $f_X(u)$ for all u.

FA15 PROBLEM 2

2. **[22 points]** Let N_t be a Poisson process with rate $\lambda > 0$.

(a) (4 points) Obtain $P\{N_3 = 5\}$.

(b) (6 points) Obtain $P\{N_7 - N_4 = 5\}$ and $E[N_7 - N_4]$.

(c) (6 points) Obtain $P\{N_7 - N_4 = 5 | N_6 - N_4 = 2\}$.

(d) (6 points) Obtain $P\{N_6 - N_4 = 2 | N_7 - N_4 = 5\}$.

FA14 PROBLEM 3

3. **[12 points]** Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ.

(a) (6 points) Express $E[N(t)N(t+s)],\ s,t > 0$ as a function of λ, s, and t.

(b) (6 points) Let $\lambda = 2$ arrivals/hour and assume that the Poisson process models the arrival of customers into a post office. Find the probability of the following event, which involves three conditions:

(Three customers arrive between 1 and 3pm,
one customer arrives between 2 and 3pm,
and one customer arrives between 2 and 4pm.)

FA15 PROBLEM 6 (MIDTERM 1)

6. **[20 points]** Consider a random variable X uniformly distributed on the set $\{1, \ldots, n\} \cup \{2n+1, \ldots, 3n\}$, i.e. $P\{X = k\}$ is constant for $k = 1, \ldots, n, 2n+1, \ldots, 3n$.

   (a) Suppose that n is unknown but it is observed that $X = 9$. Obtain the maximum likelihood estimate of n.

   (b) Suppose now that it is known that n can have two different known values, n_1 and n_2, which gives rise to two hypotheses

   $$H_0 : X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\},$$

   $$H_1 : X \in \{1, ..., n_2\} \cup \{2n_2 + 1, ..., 3n_2\},$$

   where $n_1 < n_2 < 2n_1$. Obtain the maximum likelihood decision rule.

   (c) Using the decision rule and distributions from part (b), obtain $p_{\text{false alarm}}$.

SP15 PROBLEM 4

4. [**16 points**] The two parts of this problem are unrelated.

   (a) Let X be uniformly distributed in $[0, 1]$. Find the CDF for

$$Y = 2|X - 1/2|.$$

FA15 PROBLEM 4

4. **[20 points]** Let the joint pdf for the pair (X, Y) be

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

for some constant c.

(a) (5 points) Compute the marginal $f_X(x)$. You can leave it in terms of c.

(b) (6 points) Obtain the value of the constant c for $f_{X,Y}$ to be a valid joint pdf.

(c) (5 points) Obtain $P\left\{X + Y < \frac{1}{2}\right\}$.

(d) (4 points) Are X and Y independent? Explain why or why not.

FA12 PROBLEM 6

6. **[24 points]** Suppose random variables X and Y have the joint probability density function (pdf): $f_{XY}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{elsewhere} \end{cases}$

(a) (4 points) Are X and Y independent? Explain your answer.

(b) (6 points) Determine the marginal pdf of X, $f_X(u)$.

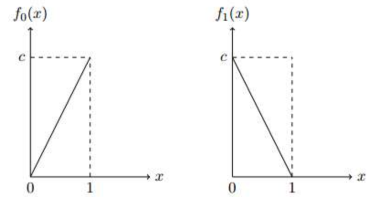
(c) (3 points) For what values of u is the conditional pdf of Y given $X = u$, $f_{Y|X}(v|u)$, well defined?

(d) (4 points) Determine $f_{Y|X}(v|u)$ for the values of u for which it is well defined. Be sure to indicate where its value is zero.

(e) (7 points) Determine $P\{Y > X\}$.

FA15 PROBLEM 3

3. **[20 points]** Let X be a continuous-type random variable taking values in $[0, 1]$. Under hypothesis H_0, the pdf of X is f_0; under hypothesis H_1, the pdf of X is f_1. Both pdfs are plotted below. The priors are known to be $\pi_0 = 0.6$ and $\pi_1 = 0.4$.



(a) (4 points) Find the value of c.

(b) (8 points) Specify the maximum a posteriori (MAP) decision rule for testing H_0 vs. H_1.

(c) (8 points) Find the error probabilities $p_{\text{false alarm}}$, $p_{\text{miss detection}}$ and the average probability of error p_e for the MAP rule.