HKN ECE 313 EXAM 2 REVIEW SESSION

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• Continuous-type Random Variables (mean and variance of CRVs)
• Uniform Distribution
• Exponential Distribution
• Poisson Process
• Linear Scaling of PDFs
• Gaussian Distribution
• ML Parameter Estimation for Continuous Random Variables
• Functions of a random variable
• Failure Rate Functions
• Binary Hypothesis Testing
• Joint CDFs, PMFs, and PDFs
• Independence of Random Variables
• Distributions of sums of random variables
CONTINUOUS-TYPE RANDOM VARIABLES

• Cumulative Distribution Functions (CDFs)
  • Must be nondecreasing
  • Must go to 0 at –infinity and 1 at +infinity
  • Must be right continuous
• $F_X(c) = \int_{-\infty}^{c} f_X(u)du$
• $P\{X = a\} = 0$
• $P\{a < X \leq b\} = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)du$
• $E[X] = \mu_X = \int_{-\infty}^{+\infty} u f_X(u)du$
UNIFORM DISTRIBUTION

• Uniform(a,b):

• pdf: \( f(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{else} \end{cases} \)

• mean: \( \frac{a+b}{2} \)

• variance: \( \frac{(b-a)^2}{12} \)
EXPONENTIAL DISTRIBUTION

- Exponential($\lambda$): limit of scaled geometric random variables
- $pdf: f(t) = \lambda e^{-\lambda t} \; t \geq 0$
- mean: $\frac{1}{\lambda}$
- variance: $\frac{1}{\lambda^2}$
- Memoryless Property
  - $P\{T \geq s + t \mid T \geq s\} = P\{T \geq t\} \; s, t \geq 0$
POISSON PROCESS

- Poisson process is used to model the number of counts in a time interval. Similar to how the exponential distribution is a limit of the geometric distribution, the Poisson Process is the limited of the Bernoulli process.

- *If we have a rate λ and time interval t, the number of counts ~Poisson(λt)*

- Disjoint intervals, e.g. 0 to 2s and 2 to 3s, are independent
LINEAR SCALING OF PDFS

- If $Y = aX + b$:
- $E[Y] = aE[X] + b$
- $Var(Y) = a^2Var(X)$
- $f_y(v) = f_x\left(\frac{v-b}{a}\right) \frac{1}{a}$
GAUSSIAN DISTRIBUTION

- Gaussian (or Normal) Distribution $\sim N(\mu, \sigma^2)$
  - Standard Gaussian, $\hat{X}: \mu = 0, \sigma = 1$
- pdf: $f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}$
- mean: $\mu$
- variance: $\sigma^2$
- If we standardize our Gaussian, where: $\hat{X} = \frac{X-\mu}{\sigma}$
  - $\Phi(c) = \int_{-\infty}^{c} f(u)du$
  - $Q(c) = 1 - \Phi(c) = \int_{c}^{\infty} f(u)du$
• Suppose we have a random variable with a given distribution/pdf that depends on a parameter, \( \theta \). By taking trials of the random variable, we can estimate \( \theta \) by finding the value that maximizes the likelihood of the observed event, \( \hat{\theta}_{ML} \).

• There are a few ways we can find \( \hat{\theta}_{ML} \)
  • Take derivative of provided pmf and set it equal to zero (maximization)
  • Observe the intervals where the likelihood increases and decreases, and find the maximum between these intervals
  • Intuition!
FUNCTIONS OF RANDOM VARIABLES

- Suppose $Y = g(X)$, and we want to be able to describe the distribution of $Y$
- Step 1: Identify the support of $X$. Sketch the pdf of $X$ and $g$. Identify the support of $Y$. Determine whether $Y$ is a Continuous or Discrete RV
- Step 2 (for CRV): Use the definition of the CDF to find the CDF of $Y$:
  - $F_Y(c) = P\{Y \leq c\} = P\{g(X) \leq c\}$
- Step 2 (for DRV): Find the pmf of $Y$ directly using the definition of the pmf
  - $p_Y(v) = P\{Y = v\} = P\{g(X) = v\}$
- Step 3 (for CRV): Differentiate the CDF of $Y$ in order to find the pdf of $Y$
GENERATING A RV WITH A SPECIFIED DISTRIBUTION

- We can generate any distribution by applying a function to a uniform distribution.

- This function should be the inverse of the CDF of the desired distribution.

- Ex: if we want an exponential distribution,
  - \( F_X(c) = 1 - e^{-c} = u \); then find \( F_X^{-1}(c) \).
FAILURE RATE FUNCTIONS

• We can assess the probability of a failure in a system through a failure rate function, \( h(t) \).

\[
F(t) = 1 - e^{-\int_0^t h(s)ds}
\]

• Two popular failure rate functions:
  • Consistent lifetime
  • “Bath tub”
BINARY HYPOTHESIS TESTING

- Similar to BHT with Discrete RVs
- Maximum Likelihood (ML) Rule
  \[ \Lambda(k) = \frac{f_1(k)}{f_0(k)} \]
  \[ \Lambda(k) = \begin{cases} > 1 & \text{declare } H_1 \text{ is true} \\ < 1 & \text{declare } H_0 \text{ is true} \end{cases} \]
- Maximum a Posteriori (MAP) Rule
  - Prior probabilities: \( \pi_0 = P(H_0), \pi_1 = P(H_1) \)
  - \( H_1 \text{ true if } \pi_1 f_1(k) > \pi_0 f_0(k) \), same as \( \Lambda(k) = \frac{f_1(k)}{f_0(k)} > \tau \) where \( \tau = \frac{\pi_0}{\pi_1} \)
- Probabilities of False Alarm and Miss
  - \( p_{false \ \text{alarm}} = P(\text{Say } H_1 \mid H_0 \text{ is true}) \)
  - \( p_{miss} = P(\text{Say } H_0 \mid H_1 \text{ is true}) \)
  - \( P_e = \pi_1 p_{\text{miss}} + \pi_0 p_{false \ \text{alarm}} \)
JOINT CDF, PMF, AND PDF

DISCRETE RANDOM VARIABLES

- Joint CDF
  \[ F_{X,Y}(u_0, v_0) = P\{X \leq u_0, Y \leq v_0\} \]
- Joint PMF
  \[ p_{X,Y}(u_0, v_0) = P\{X = u_0, Y = v_0\} \]
- Marginal PMFs
  \[ p_X(u) = \sum_j p_{X,Y}(u, v_j) \]
  \[ p_Y(v) = \sum_i p_{X,Y}(u_i, v) \]
- Conditional PMFs
  \[ p(Y|X)(v|u_0) = P(Y = v|X = u_0) = \frac{p_{X,Y}(u_0,v)}{p_X(u_0)} \]

CONTINUOUS RANDOM VARIABLES

- Joint CDF
  \[ F_{X,Y}(u_0, v_0) = P\{X \leq u_0, Y \leq v_0\} \]
- Joint PDF
  \[ f_{X,Y}(u_0, v_0) = \iint f_{X,Y}(u, v)dv \]
- Marginal PDFs
  \[ f_X(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)dv \]
  \[ f_Y(v) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)du, \]
- Conditional PDFs
  \[ f(Y|X)(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)} \]
• We can check independence in a joint distribution in a couple ways:

• \( f_{XY}(u, v) = f_X(u)f_Y(v) \)

• The support of \( f_{XY}(u, v) \) is a product set
  • Product set must have the swap property, which is satisfied if:
    • \((a, b) and (c, d) \in \text{support}(f_{XY})\), and then \((a, d) and (b, c) \in \text{support}(f_{XY})\)
  • Checking for a product set is only sufficient to prove dependence. Saying that the joint pdf is a product set is not sufficient to check independence
DISTRIBUTION OF SUMS OF RANDOM VARIABLES

- Suppose we want to find the distribution from the sum of two independent random variables where $Z = X + Y$

- The pdf or pmf of $Z$ is the convolution of the two pdfs/pdfs
5. **(18 points)** Suppose we are testing if a transistor in a circuit is working properly or not. The voltage we observe at the output is a normal random variable $X$. If the transistor is working, $X$ is distributed according to $N(1, 1)$. If the transistor is not working, then $X$ is distributed according to $N(-1, 1)$. There is a 50% chance that the transistor is working. You can express the answers for this problem in terms of $\Phi$ and $Q$.

(a) (6 points) Find $P\{X \leq 1\}$ [transistor is working].

(b) (6 points) Find $P\{X \geq 1\}$.

(c) (6 points) Obtain the unconditional pdf of $X$, $f_X(u)$ for all $u$. 
2. [22 points] Let $N_t$ be a Poisson process with rate $\lambda > 0$.
   (a) (4 points) Obtain $P\{N_5 = 5\}$.

   (b) (6 points) Obtain $P\{N_7 - N_4 = 5\}$ and $E[N_7 - N_4]$.

   (c) (6 points) Obtain $P\{N_7 - N_4 = 5|N_6 - N_4 = 2\}$.

   (d) (6 points) Obtain $P\{N_6 - N_4 = 2|N_7 - N_4 = 5\}$. 
3. [12 points] Let \( \{N(t), \ t \geq 0\} \) be a Poisson process with rate \( \lambda \).

(a) (6 points) Express \( E[N(t)N(t+s)], \ s, t > 0 \) as a function of \( \lambda, s, \) and \( t \).

(b) (6 points) Let \( \lambda = 2 \) arrivals/hour and assume that the Poisson process models the arrival of customers into a post office. Find the probability of the following event, which involves three conditions:
   (Three customers arrive between 1 and 3pm,
    one customer arrives between 2 and 3pm,
    and one customer arrives between 2 and 4pm.)
6. [20 points] Consider a random variable $X$ uniformly distributed on the set $\{1, \ldots, n\} \cup \{2n + 1, \ldots, 3n\}$, i.e. $P[X = k]$ is constant for $k = 1, \ldots, n, 2n + 1, \ldots, 3n$.

(a) Suppose that $n$ is unknown but it is observed that $X = 9$. Obtain the maximum likelihood estimate of $n$.

(b) Suppose now that $n$ can have two different known values, $n_1$ and $n_2$, which gives rise to two hypotheses:

$H_0 : X \in \{1, \ldots, n_1\} \cup \{2n_1 + 1, \ldots, 3n_1\}$,

$H_1 : X \in \{1, \ldots, n_2\} \cup \{2n_2 + 1, \ldots, 3n_2\}$,

where $n_1 < n_2 < 2n_1$. Obtain the maximum likelihood decision rule.

(c) Using the decision rule and distributions from part (b), obtain $p$ as above.
4. [16 points] The two parts of this problem are unrelated.
   
   (a) Let $X$ be uniformly distributed in $[0, 1]$. Find the CDF for $Y = 2|X - 1/2|$. 
4. [20 points] Let the joint pdf for the pair \((X, Y)\) be

\[
f_{X,Y}(x, y) = \begin{cases} \text{cxy}, & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}
\]

for some constant \(c\).

(a) (5 points) Compute the marginal \(f_X(x)\). You can leave it in terms of \(c\).

(b) (6 points) Obtain the value of the constant \(c\) for \(f_{X,Y}\) to be a valid joint pdf.

(c) (5 points) Obtain \(P\{X + Y < \frac{1}{2}\}\).

(d) (4 points) Are \(X\) and \(Y\) independent? Explain why or why not.
6. [24 points] Suppose random variables $X$ and $Y$ have the joint probability density function
(pdf): $f_{XY}(u,v) = \begin{cases} \frac{3}{2}, & u > 0, \ u^2 < v < 1 \\ 0, & \text{elsewhere} \end{cases}$

(a) (4 points) Are $X$ and $Y$ independent? Explain your answer.
(b) (6 points) Determine the marginal pdf of $X$, $f_X(u)$.
(c) (3 points) For what values of $u$ is the conditional pdf of $Y$ given $X = u$, $f_{Y|X}(v|u)$, well-defined?
(d) (4 points) Determine $f_{Y|X}(v|u)$ for the values of $u$ for which it is well defined. Be sure to indicate where its value is zero.
(e) (7 points) Determine $P\{Y > X\}$. 

3. [20 points] Let $X$ be a continuous-type random variable taking values in $[0, 1]$. Under hypothesis $H_0$, the pdf of $X$ is $f_0$; under hypothesis $H_1$, the pdf of $X$ is $f_1$. Both pdfs are plotted below. The priors are known to be $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

(a) [4 points] Find the value of $c$.

(b) [8 points] Specify the maximum a posteriori (MAP) decision rule for testing $H_0$ vs. $H_1$.

(c) [8 points] Find the error probabilities $p_{\text{false}}$ and $p_{\text{false}}$ for the MAP rule.