Topics

- Fourier Transform
- Signal Energy and Bandwidth
- LTI System Response with Fourier Transform
- Modulation, AM, Coherent Demodulation
- Impulse Response and Convolution
- Sampling and Analog Reconstruction
- LTIC and BIBO Stability
Fourier Transform

- The Fourier Transform of a signal shows the frequency content of that signal
  - In other words, we can see how much power is contained at each frequency for that signal
  - This is a big deal!
  - This is the biggest deal!

- $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$
Important Signals for Fourier Transform

- $rect\left(\frac{t}{T}\right) = \begin{cases} 1, & \text{for } |t| < \frac{T}{2} \\ 0, & \text{for } |t| > \frac{T}{2} \end{cases}$

- $u(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t > 0 \end{cases}$

- $\Delta\left(\frac{t}{T}\right) = \begin{cases} 1 + \frac{2t}{T}, & \text{for } -\frac{T}{2} < t < 0 \\ 1 - \frac{2t}{T}, & \text{for } 0 < t < \frac{T}{2} \end{cases}$

- $sinc(t) = \begin{cases} \frac{\sin(t)}{t}, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0 \end{cases}$
Fourier Transform Tips

- Convolution in the time domain is multiplication in the frequency domain
  - Conversely, multiplication in the time domain is convolution in the frequency domain

- Scaling your signal can force properties to appear; typically time delay
  - Ex: $e^{-2(t+1)}u(t - 1) \rightarrow e^{-4}e^{-2(t-1)}u(t - 1)$

- The properties really do matter! Take the time to acquaint yourself with them.

- Remember that the Fourier Transform is linear, so you can express a spectrum as the addition of two easier spectra
  - Ex: Staircase function

- Magnitude Spectrum is even symmetric, Angle Spectrum is odd symmetric for real valued signals
Signal Energy and Bandwidth

- **Energy** = \( W = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \)
  - Energy signals can be either low-pass or band-pass signals

- Bandwidth for Low-pass Signals
  - 3dB BW
    - \( \frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2} \)
  - r% BW
    - \( \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega = rW \)

- Bandwidth for Band-pass signals
  - r% BW
    - \( \frac{1}{\pi} \int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega = rW \) notice that r% BW for Band-pass signals is twice that of low-pass signals!
LTI System Response using Fourier Transform

▪ Given the following LTI system:

\[ f(t) \rightarrow H(\omega) \rightarrow y(t) \]

▪ \[ Y(\omega) = F(\omega)H(\omega) \]

▪ Computationally speaking, and in future courses, we prefer to use Fourier Transforms instead of convolution to evaluate an LTI system

▪ Why?
  ▪ So much faster, minimal error
Modulation, AM Radio, Coherent Demodulation

- Modulation Property
  - If \( f(t) \leftrightarrow F(\omega) \), \( f(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2} [F(\omega - \omega_0) + F(\omega + \omega_0)] \)

- In general, for Amplitude Modulation in communications, we modulate with cosine in order to shift our frequency spectrum into different frequency bands.

- Coherent Demodulation refers to the process of modulating a signal in order to make it band-pass, then modulating it back to the original low-pass baseband before low-pass filtering in order to recover the original signal.

- Envelope detection is the process of Full-wave Rectification (absolute value), then low-pass filtering in order to extract the signal.
Impulse Response and Convolution

- Convolution
  \[ x(t) \ast y(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau \]
  - We “flip and shift” one signal and evaluate the integral of the product of the two signals at any value of \( t \) (our delay for the shift)

- Representing LTI Systems
  - \( y(t) = x(t) \ast h(t) \), where \( h(t) \) is the impulse response of the system

- Impulse Response is the system output to a \( \delta(t) \) input

- Graphical convolution helps to visualize the process of flipping and shifting
Helpful Properties for Convolution

• Derivative
  - $h(t) * f(t) = y(t) \rightarrow \frac{d}{dt} h(t) * f(t) = h(t) * \frac{d}{dt} f(t) = \frac{d}{dt} y(t)$
  - Use of Derivative property: Finding the impulse response from the unit-step response
    - If $y(t) = u(t) * h(t)$, then $\frac{d}{dt} y(t) = \frac{d}{dt} u(t) * h(t) = \delta(t) * h(t) = h(t)$

• Start Point
  - If the two signals have start points at $t_1$ and $t_2$, then the start point of their convolution will be at $t_1 + t_2$

• End Point
  - Similarly for the end points, if the two signals have end points at $t_1$ and $t_2$, then the end point of their convolution will be at $t_1 + t_2$

• Width
  - From the above two properties, we can see that if the two signals have widths $W_1$ and $W_2$, then the width of their convolution will be $W_1 + W_2$
The Impulse Function $\delta(t)$

- The impulse function is the limit of $\frac{1}{T} \text{rect} \left( \frac{t}{T} \right)$ as $T \to 0$
  - Infinitesimal Width
  - Infinite Height
  - Of course, it integrates to 1. ($0 \times \infty = 1$)

- Sifting
  - $\int_{a}^{b} \delta(t - t_o) f(t) \, dt = f(t_o)$ if $t_o$ lies in your limits of integrations; 0 else

- Sampling
  - $f(t)\delta(t - t_o) = f(t_o)\delta(t - t_o)$

- Unit-step derivative
  - $\frac{du}{dt} = \delta(t)$
Sampling and Analog Reconstruction

- If we have an original analog signal $f(t)$
- Our digital samples of the signal are obtained through sampling property as:
  - $f[n] = f(nT)$ where $T$ is our sampling period; this is Analog to Digital (A/D) conversion
- We must make sure to satisfy Nyquist Criterion:
  - $T < \frac{1}{2B}$ or $f_s > 2B$
  - To learn more about Nyquist Criterion take ECE 110!
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Sampling and Analog Reconstruction

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- Our digital samples of the signal are obtained through sampling property as:
  - \( f[n] = f(nT) \) where \( T \) is our sampling period; this is Analog to Digital (A/D) conversion
  - This results in infinitely many copies of the original signal’s Fourier Transform spaced by \( \frac{2\pi}{T} \)

- We must make sure to satisfy Nyquist Criterion:
  - \( T < \frac{1}{2B} \) or \( f_s > 2B \)
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- Following A/D conversion, we perform D/A conversion, then low-pass filter our signal in order to obtain our original signal according to the following relation

  - \( f(t) = \sum_n f_n sinc\left(\frac{\pi}{T} (t - nT')\right) \)

- For a more complete explanation, take ECE 310!
**LTIC**

- LTIC stands for Linearity Time-Invariance and Causality (sometimes called LSIC, where SI means Shift-Invariance)

- **Linearity**
  - Satisfy Homogeneity and Additivity
  - Can be summarized by Superposition
    - If \( x_1(t) \rightarrow y_1(t) \) and \( x_2(t) \rightarrow y_2(t) \), then \( ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t) \)

- **Shift Invariance**
  - If \( x(t) \rightarrow y(t) \) then \( x(t - t_o) \rightarrow y(t - t_o) \) \( \forall t_0 \) and \( x(t) \)

- **Causality**
  - Output cannot depend on future input values
BIBO Stability

- A system is BIBO stable if for any bounded input, we obtain a bounded output
- Two ways to check BIBO stability
  - By the definition:
    - If $|f(t)| \leq \alpha < \infty$, then $|y(t)| \leq \beta < \infty \ \forall \ t$
  - By Absolute Integrability
    - $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$
Problem 1 FA13

1. (10 pts) Let $f(t)$, plotted below, have Fourier transform $F(\omega)$.

(a) (4 pts) Obtain the Fourier transform of $x(t)$, plotted below, in terms of $F(\omega)$.

(b) (6 pts) Obtain the Fourier transform of $z(t)$, plotted below, in terms of $F(\omega)$. 
Problem 5 FA13

5. (8 pts) The two parts of this problem deal with impulse response, causality and BIBO stability.

(a) (4 pts) The figure below depicts a LTI system. Input signal \( f(t) \) produces the corresponding output \( y(t) \).

(b) (4 pts) The figure below depicts a LTI system. Input signal \( f(t) \) produces the corresponding output \( y(t) \).

a)i. Determine \( h(t) \)
ii. Is the system causal?
iii. Is the system BIBO stable?
b)i. Determine \( h(t) \)
ii. Is the system causal?
iii. Is the system BIBO stable?
Problem 2 SP16

(i) Consider this sampling/reconstruction system.

\[ f(t) = 2 \text{sinc}^2(2t) \rightarrow s(t) \rightarrow y(t) \]

\[ F(\omega) \rightarrow \sum_{n=-\infty}^{\infty} \delta(t - n \frac{\pi}{2}) \]

(i) Circle the correct answer and explain it below.

\[ f(t) \text{ is: UNDERSAMPLED} / \text{OVERSAMPLED} / \text{SAMPLED AT NYQUIST RATE} \]

Explanation:

(ii) Sketch \( S(\omega) \) and \( Y(\omega) \) on the axes below.

(iii) Determine \( y(t) \)
Problem 2 SP16

(a) For \( h(t) \) and \( f(t) \) shown below, compute the specified values for \( y(t) = f(t) \ast h(t) \)

\[ \begin{align*}
\ \ \ \ \ \ \ \ \ \ f(t) \\
1 & \quad 1 \quad 2
\end{align*} \]

\[ \begin{align*}
\ \ \ \ \ \ \ \ \ \ h(t) \\
1 & \quad 2
\end{align*} \]

\[
\begin{align*}
y(-0.5) &= \\
y(0.5) &= \\
y(1.5) &= \\
y(2.5) &= \\
y(3.5) &= \\
y(4.5) &= 
\end{align*}
\]

(b) Let \( h(t) = e^{-t}u(t) \) and \( f(t) = u(t-4) \). Find \( y(t) = f(t) \ast h(t) \) for all values of \( t \).
Problem 3 SP14

(a) An impulse response is given by

\[ h(t) = \text{rect} \left( \frac{t - 1}{2} \right) e^{-t} \]

i) Find the Fourier transform of \( h(t) \).

ii) If the input is \( f(t) = 2 \text{rect} \left( \frac{t}{2} \right) \), find the output \( y(t) = f(t) \ast h(t) \).

iii) If the input is \( f(t) = \frac{1}{a} \text{rect} \left( at \right) \), find the output \( y(t) = f(t) \ast h(t) \), when \( a \to 0 \).
Problem 1 FA15

a) Find the Fourier transform \( F(\omega) \) and the energy of the following signal \( f(t) \).
\[
f(t) = [u(t-1) - u(t-3)] e^{-t}
\]

b) Fourier transform of a signal \( f(t) \) is given as

\[
|F(\omega)|
\]
\[
\frac{\angle F(\omega)}{\omega}
\]

i) Find the inverse Fourier transform \( f(t) \). Simplify your answer. Sketch \( f(t) \), label axis carefully.

\[
f(t) = \text{__________________________} (7 \text{ points})
\]

ii) Find the 90% energy bandwidth of \( f(t) \).
Problem 2 FA15

Consider the following system

\[ f(t) \xrightarrow{H_1(\omega)} (a) \xrightarrow{\cos \omega_c t} (b) \xrightarrow{H_1(\omega)} y(t) \]

where \( f(t) = \text{rect} \left( \frac{t}{\frac{1}{2}} \right), \tau = 10^{-3} \text{ sec}, \omega_c = 2\pi \times 10^6 \text{ rad/sec} \) and

\[ H_1(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq 2\pi \times 10^3 \\ 0 & \text{otherwise} \end{cases} \]

a) What is \( F(\omega) \)? Label axis carefully.

\[ F(\omega) \]

\[ F(\omega) = \text{___________} \text{(5 points)} \]

b) Draw the signal spectrum at points, (a), (b) and (c). Label axis carefully.
Problem 1 SP16

- Consider a real-valued function $f(t)$ with bandwidth $\Omega$ and let $\omega_c > \Omega$. Obtain the bandwidth of the following functions. All answers may be left in terms of $\Omega$ and $\omega_c$.

- (a) $g_1(t) = f(t) \sin(\omega_c t)$
- (b) $g_2(t) = f(t) + \sin(\omega_c t)$
- (c) $g_3(t) = f(t) \sin^2(\omega_c t)$
- (d) $g_4(t) = f^2(t) \sin(\omega_c t)$
- (e) $g_5(t) = f(t) * \sin(\omega_c t)$
Problem 2 SP16

(a) Let $y_1(t) = 2tu(t)$. Obtain and simplify $\frac{dy_1(t)}{dt}$.

\[
\frac{dy_1(t)}{dt} =
\]

(b) Let $f_2(t) = e^{-2u(t)}$. Obtain and simplify $y_2(t) = \int_{-\infty}^{5} f_2(\tau)\delta(4 + \tau)d\tau$.

\[
y_2(t) =
\]

(c) Let $f_3(t) = u(t-1)$ and $y_3(t) = f_3(t) \ast h_3(t) = rect \left( \frac{t}{2} - 3 \right)$. Obtain and simplify $h_3(t)$.