Chapter 3
Circuit for Signal Processing
3.1 op-amp & signal arithmetic

• Four inputs: terminals with node voltage V+ and V-; terminals with biasing voltage $V_b$ and $-V_b$ ($V_b$ is positive)
• One output: terminal with node voltage $V_o$
\[ V_o = A(V_+ - V_-) - R_o i_o \]
• $V_o = A(V_+ - V_-) - R_o i_o$
• $|V_o|$ cannot be larger than $V_b$
\( R_o \sim 1 - 10 \, \Omega \)
\( R_i \sim 10^6 \, \Omega \)
\( A \sim 10^6 \)

\( R_o = 0 \, \Omega \)
\( R_i \to \infty \, \Omega \)
\( A \to \infty \)
$R_o \sim 1 - 10 \, \Omega$

$R_i \sim 10^6 \, \Omega$

$A \sim 10^6$

$R_o = 0 \, \Omega$

$R_i \rightarrow \infty \, \Omega$

$A \rightarrow \infty$
• $i_+ = i_- = 0$
• $V_+ = V_-$
• Voltage follower
• Voltage follower

• $V_+ = V_-$
• $V_{in} = V_{out}$

• The output side circuit won’t affect the voltage output.
- Noninverting amplifier
• Noninverting amplifier

• \( V^+ = V^- = V_{in} \)

• \( i^- = 0 \)

• According to KCL,
  \[
  \frac{V_-}{R_1} = \frac{V_{out} - V_-}{R_2}
  \]

• \( V_{out} = \left( 1 + \frac{R_2}{R_1} \right)V_{in} \)
• Noninverting amplifier

• Negative feedback
Connecting the output of an op-amp to its inverting (-) input is called *negative feedback.*
• Why negative feedback?
• When the output of an op-amp is directly connected to its inverting (-) input, a voltage follower will be created. An op-amp with negative feedback will try to drive its output voltage to whatever level necessary so that the differential voltage between the two inputs is practically zero.

\[ V_o = A(V_+ - V_-) - R_o i_o \]
• Inverting amplifier
• Inverting amplifier

\[ i_1 + i_2 = 0 \quad V^+ = V^- = 0 \]

\[ \frac{V_{\text{in}}}{R_1} + \frac{V_{\text{out}}}{R_2} = 0 \]

\[ V_{\text{out}} = -\frac{R_2}{R_1} V_{\text{in}} \]

• Voltage gain

\[ G = \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1} \]
• Sum and difference calculators

sum
difference
3.2 Differentiators and Integrators

• Capacitor and inductor

Capacitor
• $i(t) = C \frac{dv(t)}{dt}$
• $V(t) = \left( \int_{t_0}^{t} \frac{i(t)}{C} \right) + V(t_0)$

Inductor
• $V(t) = L \frac{di(t)}{dt}$
• $i(t) = \left( \int_{t_0}^{t} \frac{V(t)}{L} \right) + i(t_0)$
\( V(t) = u(t) \)

"unhealthy" differentiator circuit.

\( i(t) = u(t) \)

integrator

Capacitor

- \( i(t) = C \frac{dv(t)}{dt} \)
- \( V(t) = \left( \int_{t_0}^{t} \frac{i(t)}{C} \right) + V(t_0) \)
\[ V(t) = L \frac{di(t)}{dt} \]

\[ i(t) = \left( \int_{t_0}^{t} \frac{V(t)}{L} \right) + i(t_0) \]

"unhealthy" differentiator circuit.

integrator
• Op-amp differentiator
\( V_- = V_+ = 0 \)

\( i_s = C \frac{d(V_s - V_-)}{dt} = C \frac{dV_s}{dt} \)

\( i_s = i_r \)

\( V_o = V_- - i_r R = -RC \frac{dV_s}{dt} \)

Capacitor

\[ i(t) = C \frac{dv(t)}{dt} \]

\[ V(t) = (\int_{t_0}^{t} \frac{i(t)}{C} dt) + V(t_0) \]
\begin{itemize}
\item $V_- = V_+ = 0$
\item $i_r = \frac{V_s}{R}$
\item $i_L = i_r$
\item $V_- - V_o = L \frac{di_L}{dt}$
\item $V_o = - \frac{L}{R} \frac{dV_s}{dt}$
\end{itemize}

Inductor
\[ V(t) = L \frac{di(t)}{dt} \]
\[ i(t) = (\int_{t_0}^{t} \frac{V(t)}{L}) + i(t_0) \]
• Op-amp integrator
\[ V_- = V_+ = 0 \]
\[ i_r = \frac{V_s}{R} \]
\[ i_c = i_r \]
\[ V_- - V_o = (\int_{t_0}^{t} \frac{i_c(t)}{C}) + V_c(t_0) \]
\[ V_o = -\left(\int_{t_0}^{t} \frac{V_s(t)}{RC}\right) - V_c(t_0) \]

\[ i(t) = C \frac{dv(t)}{dt} \]
\[ V(t) = \left(\int_{t_0}^{t} \frac{i(t)}{C}\right) + V(t_0) \]
• $V_- = V_+ = 0$

• $i_L = \left( \int_{t_0}^{t} \frac{V_s(t)}{L} \right) + i(t_0)$

• $i_L = i_r$

• $V_- - V_o = i_r R = \left( \left( \int_{t_0}^{t} \frac{V_s(t)}{L} \right) + i(t_0) \right) R$

• $V_o = -\left( \left( \int_{t_0}^{t} \frac{V_s(t)}{L} \right) + i(t_0) \right) R$

\[
V(t) = L \frac{di(t)}{dt}
\]

\[
i(t) = \left( \int_{t_0}^{t} \frac{V(t)}{L} \right) + i(t_0)
\]
3.3 Linearity, Time Invariance, & LTI System

- **Linearity**

  \[ y(t) = y(t_0) + \int_{t_0}^{t} f(\tau) d\tau \]

  - Zero-input response
  - Zero-state response

- A system is said to be linear if its output is a sum of distinct zero-input and zero-state responses that vary linearly with the initial state of the system and linearly with the system input, respectively.
• How to decide whether linear or not?
• Assume
  • \( f_1(t) \rightarrow y_1(t) \)
  • \( f_2(t) \rightarrow y_2(t) \)
  • \( f_3(t) = C_1f_1(t) + C_2f_2(t) \)
• If \( y_3(t) = C_1y_1(t) + C_2y_2(t) \) holds true, then we say the system is linear
• E.g. Verify $y(t) = \int_{t_0}^{t} f(t)dt$ is linear.

• Assume
  • $f_1(t) \to y_1(t), y_1(t) = \int_{t_0}^{t} f_1(t)dt$
  • $f_2(t) \to y_2(t), y_2(t) = \int_{t_0}^{t} f_2(t)dt$
  • $f_3(t) = C_1 f_1(t) + C_2 f_2(t)$
  • $y_3(t) = \int_{t_0}^{t} f_3(t)dt = \int_{t_0}^{t} (C_1 f_1(t) + C_2 f_2(t))dt$

\[= C_1 \int_{t_0}^{t} f_1(t)dt + C_2 \int_{t_0}^{t} f_2(t)dt\]
\[= C_1 y_1(t) + C_2 y_2(t)\]

• That this system is linear is hereby proved.
• E.g. Verify \( y(t) = f^2(t) \) is not linear.

• Assume
  • \( f_1(t) \rightarrow y_1(t), y_1(t) = f_1^2(t) \)
  • \( f_2(t) \rightarrow y_2(t), y_2(t) = f_2^2(t) \)
  • \( f_3(t) = c_1f_1(t) + c_2f_2(t) \)

• \( y_3(t) = f_3^2(t) = (c_1f_1(t) + c_2f_2(t))^2 \neq c_1y_1(t) + c_2y_2(t) \)

• That this system is not linear is hereby proved.
- Time-invariance
- Delayed inputs cause equally delayed outputs
• How to decide whether time-invariant or not?

• Assume
  • \( f_1(t) \rightarrow y_1(t) \)
  • \( f_2(t) = f_1(t - t_0) \)
  • \( f_2(t) \rightarrow y_2(t) \)

• If \( y_2(t) = y_1(t - t_0) \) holds true, then we say the system is time-
invariant
• E.g. Verify $y(t) = \int_{t_0}^{t} f(t)dt$ is time-invariant.

• Assume
  • $f_1(t) \to y_1(t), y_1(t) = \int_{t_0}^{t} f_1(t)$
  • $f_2(t) = f_1(t - t_0)$
  • $f_2(t) \to y_2(t)$

• $y_2(t) = \int_{t_0}^{t} f_2(t)dt = \int_{t_0}^{t} f_1(t - t_0)dt$

$y_1(t - t_0) = \int_{t_0}^{t} f_1(t - t_0)$

$y_2(t) = y_1(t - t_0)$

That this system is time-invariant is hereby proved.
• E.g. Verify $y(t) = \int_{t_0}^{t} f(3 - t)dt$ is not time-invariant.

• Assume
  
  • $f_1(t) \rightarrow y_1(t), y_1(t) = \int_{t_0}^{t} f_1(3 - t)$
  • $f_2(t) = f_1(t - t_0)$
  • $f_2(t) \rightarrow y_2(t)$

• $y_2(t) = \int_{t_0}^{t} f_2(t)dt = \int_{t_0}^{t} f_1((3 - t) - t_0)dt$

$y_1(t - t_0) = \int_{t_0}^{t} f_1(3 - (t - t_0))$

$y_2(t) \neq y_1(t - t_0)$

That this system is not time-invariant is hereby proved.
3.4 First-order RC & RL Circuits

- Capacitor and inductor

**Capacitor**
- \(i(t) = C \frac{dv(t)}{dt}\)
- \(V(t) = (\int_{t_0}^{t} \frac{i(t)}{C}) + V(t_0)\)

**Inductor**
- \(V(t) = L \frac{di(t)}{dt}\)
- \(i(t) = (\int_{t_0}^{t} \frac{V(t)}{L}) + i(t_0)\)
• RC-circuit response to constant sources
• KVL
  - $V_s = V_r + V$
• $V_s = iR + V$
• $V_s = (C \frac{dv(t)}{dt})R + V$
• $\frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} V_s$

\[ i(t) = C \frac{dv(t)}{dt} \]

\[ V(t) = \left( \int_{t_0}^{t} \frac{i(t)}{C} \right) + V(t_0) \]
\[ \frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} V_s \]

\[ v = A e^{-\frac{t}{RC}} + V_s \]

When \( t = 0 \), \( v(0) = v(0-) = A + V_s \)

\[ A = v(0-) - V_s \]
3.3 Linearity, Time Invariance, & LTI System

- **Linearity**
  \[ y(t) = y(t_0) + \int_{t_0}^{t} f(\tau)d\tau \]
  Zero-input response  Zero-state response

- A system is said to be linear if its output is a sum of distinct zero-input and zero-state responses that vary linearly with the initial state of the system and linearly with the system input, respectively.
\[
\frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} V_s
\]

- \( v = Ae^{-\frac{t}{RC}} + V_s \)
- \( A = v(0-) - V_s \)

\[ v = (v(0-) - V_s)e^{-\frac{t}{RC}} + V_s \]

\[ v = e^{-\frac{t}{RC}v(0-) + (1 - e^{-\frac{t}{RC}})V_s} \]

\[ v = v_{\text{zero-input}} + v_{\text{zero-state}} \]
• RL-circuit response to constant sources
• KCL
  - \( I_s = i_r + i \)
  - \( I_s = \frac{V}{R} + i \)
  - \( I_s = \frac{(L \frac{di(t)}{dt})}{R} + i \)
  - \( \frac{di}{dt} + \frac{R}{L} i(t) = \frac{R}{L} I_s \)

\[
V(t) = L \frac{di(t)}{dt} \\
i(t) = (\int_{t_0}^{t} \frac{V(t)}{L}) + i(t_0)
\]
• \( \frac{di}{dt} + \frac{R}{L} i(t) = \frac{R}{L} I_s \)

\[ i = (i(0^-) - I_s)e^{-\frac{t}{L/R}} + I_s \]

\[ i = e^{-\frac{t}{L/R}}i(0^-) + (1 - e^{-\frac{t}{L/R}})I_s \]

\[ i = i_{\text{zero-input}} + i_{\text{zero-state}} \]
• Initial energy

• $E_c = \frac{1}{2} CV^2$

• $E_c = \frac{1}{2} CV(0-)^2$

• $E_L = \frac{1}{2} LI^2$

• $E_L = \frac{1}{2} LI(0-)^2$
• Steady state
4. (10 pts) In the following circuit, assuming linear operation and ideal op-amp approximation, express the output voltage $V_o$, in terms of $V_1$ and $V_2$.

\[ V_o = 4V_1 - 4V_2 \]
Problem 4  (25 points)

Assume the switch has been in position A for a long time. It moves to position B at $t = 0$.

a)  (5 points)  Write the 1st order ODE of $v(t)$ for $t > 0$.

b)  (3 points)  Find the initial value of $v(t)$ at $t = 0^-$.

c)  (8 points)  Solve $v(t)$ for $t > 0$.

d)  (3 points)  What is the zero input component of $v(t)$?

e)  (3 points)  What is the zero state component of $v(t)$?

f)  (3 points)  What is the steady state value of $v(t)$ for $t > 0$?
a) (5 points) Write the 1\textsuperscript{st} order ODE of v(t) for t > 0.

\[
i(t) = C \frac{dv(t)}{dt}
\]

\[
V(t) = \left( \int_{t_0}^{t} \frac{i(t)}{C} \right) + V(t_0)
\]
b) (3 points) Find the initial value of \( v(t) \) at \( t = 0^- \).

\[
\begin{align*}
i(t) &= C \frac{dv(t)}{dt} \\
V(t) &= \left( \int_{t_0}^{t} \frac{i(t)}{C} \right) + V(t_0)
\end{align*}
\]
c) (8 points) Solve \( v(t) \) for \( t > 0 \).

- \( \frac{dv}{dt} + \frac{1}{2} v = 1 \)
- \( v_{0-} = 4 \)
- \( v = Ae^{-\frac{t}{RC}} + V_s \)
- \( A = v(0-) - V_s \)
c) (8 points) Solve $v(t)$ for $t > 0$.

\[ \frac{dv}{dt} + \frac{1}{2} v = 1 \]

\[ v_{0-} = 4 \]

\[ v = Ae^{-\frac{t}{RC}} + V_s \]

\[ A = v(0^-) - V_s \]
d) (3 points) What is the zero input component of $v(t)$?

e) (3 points) What is the zero state component of $v(t)$?

\[
\begin{align*}
\nu &= (\nu(0) - \nu_s)e^{-\frac{t}{RC}} + \nu_s \\
\nu &= e^{-\frac{t}{RC}}\nu(0) + (1 - e^{-\frac{t}{RC}})\nu_s \\
\nu &= \nu_{\text{zero-input}} + \nu_{\text{zero-state}}
\end{align*}
\]
f) (3 points) What is the steady state value of \( v(t) \) for \( t > 0 \)?
Questions?